

The Natural Deduction Pack

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1 Using this pack

This pack consists of Natural Deduction problems, intended to be used alongside *The Logic Manual* by Volker Halbach. The pack covers Natural Deduction proofs in propositional logic (\mathcal{L}_1), predicate logic (\mathcal{L}_2) and predicate logic with identity ($\mathcal{L}_=$). The vast majority of these problems ask for the construction of a Natural Deduction proof; there are also worked examples explaining in more detail the proof strategies for some connectives, as well as some questions about Natural Deduction which are more unusual.

The pack hopefully offers more questions to practice with than any student should need, but the sheer number of problems in the pack can be daunting. For this reason there is also a ‘core’ set of questions aimed at covering the most crucial skills needed to tackle a Natural Deduction proof.

Next to each problem is a number in brackets indicating the number of steps in my solution. This can be taken as a rough measure of the difficulty of the problem, although it should be emphasised that this is not always perfect: some long proofs can be methodical, while some short proofs can be counter-intuitive.

For each of these problems I provide a proof and an explanation of the strategy behind the proof. I use additional notation to annotate the Natural Deduction proofs in two ways. First, next to each horizontal line in a proof I label which rule has been applied. Where a connective has a pair of introduction rules (such as \forall Intro1 and \forall Intro2) or a pair of elimination rules (such as \wedge Intro1 and \wedge Intro2), I only distinguish between the first and second versions in the solutions for the earlier problems; in the later stages it should be clear which version has been applied. Second, for longer proofs, I sometimes use a number to mark both a discharged assumption and the point in the proof when that assumption is discharged. When an assumption has been discharged by $=$ Intro I label the assumption with a superscript $=$. These are not formal components of a proof, but they should help in explaining how the proof has been constructed.

The solutions I provide are never the only possible solutions. Usually I aim to provide the shortest possible solution, but in some cases I also present possible alternative proofs.

At the very end of the pack there are extra problems focused on ways in which Natural Deduction can be altered or extended, either by adding new rules or replacing existing rules.

If this pack is viewed as a PDF, it is possible to click on any problem to go directly to its solution. Clicking on the solution’s header will take you back to the problem list. When using a printed copy of the document, these hyperlinks are unlikely to work as intended.

A changelog for the pack and an archive of past versions are maintained at github.com/Alastair-Carr/Natural-Deduction-Pack; if you have any comments, questions or suggestions, do not hesitate to contact me there.

2 Summary of rules

Propositional logic \mathcal{L}_1

Conjunction

$$\frac{\begin{array}{c} \vdots \\ \phi \end{array} \quad \begin{array}{c} \vdots \\ \psi \end{array}}{\phi \wedge \psi} \wedge\text{Intro} \quad \frac{\begin{array}{c} \vdots \\ \phi \wedge \psi \end{array}}{\phi} \wedge\text{Elim1} \quad \frac{\begin{array}{c} \vdots \\ \phi \wedge \psi \end{array}}{\psi} \wedge\text{Elim2}$$

Implication

$$\frac{\begin{array}{c} [\phi] \\ \vdots \\ \psi \end{array}}{\phi \rightarrow \psi} \rightarrow\text{Intro} \quad \frac{\begin{array}{c} \vdots \\ \phi \end{array} \quad \begin{array}{c} \vdots \\ \phi \rightarrow \psi \end{array}}{\psi} \rightarrow\text{Elim}$$

Disjunction

$$\frac{\begin{array}{c} \vdots \\ \phi \end{array}}{\phi \vee \psi} \vee\text{Intro1} \quad \frac{\begin{array}{c} \vdots \\ \psi \end{array}}{\phi \vee \psi} \vee\text{Intro2}$$

$$\frac{\begin{array}{c} [\phi] \\ \vdots \\ \phi \vee \psi \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \chi \end{array} \quad \begin{array}{c} \vdots \\ \chi \end{array}}{\chi} \vee\text{Elim}$$

Biconditional

$$\frac{\begin{array}{c} [\phi] \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \phi \end{array}}{\phi \leftrightarrow \psi} \leftrightarrow\text{Intro}$$

$$\frac{\begin{array}{c} \vdots \\ \phi \end{array} \quad \begin{array}{c} \vdots \\ \phi \leftrightarrow \psi \end{array}}{\psi} \leftrightarrow\text{Elim1} \quad \frac{\begin{array}{c} \vdots \\ \psi \end{array} \quad \begin{array}{c} \vdots \\ \phi \leftrightarrow \psi \end{array}}{\phi} \leftrightarrow\text{Elim2}$$

Negation

$$\frac{\begin{array}{c} [\phi] \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} [\phi] \\ \vdots \\ \neg\psi \end{array}}{\neg\phi} \neg\text{Intro} \quad \frac{\begin{array}{c} [\neg\phi] \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} [\neg\phi] \\ \vdots \\ \neg\psi \end{array}}{\phi} \neg\text{Elim}$$

Predicate logic \mathcal{L}_2

Universal quantifier

$$\frac{\vdots}{\forall v \phi} \forall \text{Intro}$$

provided the constant t does not occur in ϕ or in any undischarged assumption in the proof of $\phi[t/v]$

$$\frac{\vdots}{\phi[t/v]} \forall \text{Elim}$$

Existential quantifier

$$\frac{\vdots}{\exists v \phi} \exists \text{Intro}$$

$$\frac{\begin{array}{c} [\phi[t/v]] \\ \vdots \\ \exists v \phi \end{array} \quad \begin{array}{c} \vdots \\ \psi \end{array}}{\psi} \exists \text{Elim}$$

provided the constant t does not occur in $\exists v \phi$, or in ψ , or in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ

Predicate logic with identity $\mathcal{L}_=$

$$\frac{}{[t = t]} = \text{Intro}$$

\vdots

$$\frac{\begin{array}{c} \vdots \\ \phi[s/v] \end{array} \quad \begin{array}{c} \vdots \\ s = t \end{array}}{\phi[t/v]} = \text{Elim}$$

$$\frac{\begin{array}{c} \vdots \\ \phi[s/v] \end{array} \quad \begin{array}{c} \vdots \\ t = s \end{array}}{\phi[t/v]} = \text{Elim}$$

3 Worked examples

3.1 Implication

We can use the rules for implication and conjunction to prove the following theorem:

$$\vdash (P \rightarrow Q) \rightarrow ((P \wedge R) \rightarrow (Q \wedge R))$$

The easiest way to start is by working from the bottom upwards, especially since we aren't given any premises to work from.

We know that the theorem we want to prove is an implication: it is a statement of the form $\phi \rightarrow \psi$. That means we can prove it by assuming ϕ , giving a proof of ψ and then applying \rightarrow Intro (discharging all of our assumptions of ϕ). Here, ϕ corresponds to $P \rightarrow Q$ and ψ corresponds to $(P \wedge R) \rightarrow (Q \wedge R)$, so our proof will look like this:

$$\frac{\begin{array}{c} [P \rightarrow Q] \\ \vdots \\ (P \wedge R) \rightarrow (Q \wedge R) \end{array}}{(P \rightarrow Q) \rightarrow ((P \wedge R) \rightarrow (Q \wedge R))} \rightarrow\text{Intro}$$

What is shown above isn't a proof, but a way of helping us put the proof together. We know that we can assume $P \rightarrow Q$ as many times as we like, because our final \rightarrow Intro step will discharge all our assumptions of $P \rightarrow Q$.

$(P \wedge R) \rightarrow (Q \wedge R)$ is also an implication, so we can prove it by assuming $P \wedge R$ and proving $Q \wedge R$:

$$\frac{\begin{array}{c} [P \rightarrow Q], [P \wedge R] \\ \vdots \\ Q \wedge R \end{array}}{(P \wedge R) \rightarrow (Q \wedge R)} \rightarrow\text{Intro} \\ \frac{\quad}{(P \rightarrow Q) \rightarrow ((P \wedge R) \rightarrow (Q \wedge R))} \rightarrow\text{Intro}$$

We can provide a proof of $Q \wedge R$ by providing a proof of Q and a proof of R and then applying the \wedge Intro rule:

$$\frac{\begin{array}{c} [P \rightarrow Q], [P \wedge R] \\ \vdots \\ Q \end{array} \quad \begin{array}{c} [P \rightarrow Q], [P \wedge R] \\ \vdots \\ R \end{array}}{Q \wedge R} \wedge\text{Intro} \\ \frac{\quad}{(P \wedge R) \rightarrow (Q \wedge R)} \rightarrow\text{Intro} \\ \frac{\quad}{(P \rightarrow Q) \rightarrow ((P \wedge R) \rightarrow (Q \wedge R))} \rightarrow\text{Intro}$$

Now we have two branches to consider. Note that we can use our assumptions of $P \rightarrow Q$ and $P \wedge R$ in both branches: our applications of \rightarrow Intro discharge all occurrences of $P \rightarrow Q$ and $P \wedge R$ above them in the proof.

We'll consider the right branch first, because it's the more straightforward branch. We can easily obtain R by using our assumption of $P \wedge R$ and applying \wedge Elim; we don't even need to use our assumption of $P \rightarrow Q$.

$$\frac{\begin{array}{c} [P \rightarrow Q], [P \wedge R] \\ \vdots \\ Q \quad \frac{[P \wedge R]}{R} \wedge\text{Elim} \\ \frac{Q \quad R}{Q \wedge R} \wedge\text{Intro} \\ \frac{(P \wedge R) \rightarrow (Q \wedge R)}{\rightarrow\text{Intro}} \end{array}}{(P \rightarrow Q) \rightarrow ((P \wedge R) \rightarrow (Q \wedge R))} \rightarrow\text{Intro}$$

The left branch requires two steps. Applying \wedge Elim on $P \wedge R$ gives us P . Using this P and our assumption of $P \rightarrow Q$ allows us to prove Q by \rightarrow Elim:

$$\frac{\frac{\frac{[P \wedge R]}{P} \wedge\text{Elim} \quad [P \rightarrow Q]}{Q} \rightarrow\text{Elim} \quad \frac{[P \wedge R]}{R} \wedge\text{Elim}}{\frac{Q \quad R}{Q \wedge R} \wedge\text{Intro}} \rightarrow\text{Intro}}{\frac{(P \wedge R) \rightarrow (Q \wedge R)}{\rightarrow\text{Intro}}} \rightarrow\text{Intro}$$

This gives us a complete proof.

3.2 Universal quantifier

Using the introduction and elimination rules for the universal quantifier we can construct a proof of the following:

$$\forall x \neg \forall y (Pxy \rightarrow Qxy) \vdash \forall x \neg \forall y \neg Pxy$$

Our conclusion is a universal statement, so we can prove it by applying the \forall Intro rule. In order to apply the \forall Intro rule we need to prove that $\forall x \neg \forall y (Pxy \rightarrow Qxy) \vdash \forall x \neg \forall y \neg Pxy$ is true when any arbitrary constant is substituted for x ; we will choose a for our arbitrary constant, but we need to ensure that a appears in no undischarged assumptions when we apply the \forall Intro rule. This means our proof will look like this:

$$\frac{\begin{array}{c} \forall x \neg \forall y (Pxy \rightarrow Qxy) \\ \vdots \\ \neg \forall y \neg Pay \end{array}}{\forall x \neg \forall y \neg Pxy} \forall\text{Intro}$$

How do we get from $\forall x \neg \forall y (Pxy \rightarrow Qxy)$ to $\neg \forall y Pay$? $\neg \forall y Pay$ is a negated statement, so we can prove it by assuming $\forall y Pay$ and showing it leads to a contradiction.

Where can we find a contradiction? Neither the assumption $\forall y Pay$ nor our premise $\forall x \neg \forall y (Pxy \rightarrow Qxy)$ has a negation as their main connective, but from $\forall x \neg \forall y (Pxy \rightarrow Qxy)$ we can derive (by \forall Elim) $\neg \forall y (Pay \rightarrow Qay)$, which is a negated statement. This means we can obtain a contradiction if we can somehow derive $\forall y (Pay \rightarrow Qay)$ from our assumption of $\forall y Pay$ and our premise $\forall x \neg \forall y (Pxy \rightarrow Qxy)$.

Note that the assumption $\forall y Pay$ contains a , but if we can derive a contradiction from it and successfully apply \neg Intro this assumption will be discharged before we apply \forall Intro. If it is left undischarged by the time we reach the final step of the proof, we won't be able to apply \forall Intro.

$$\begin{array}{c} \forall x \neg \forall y (Pxy \rightarrow Qxy), [\forall y \neg Pay] \\ \vdots \\ \frac{\forall y (Pay \rightarrow Qay) \quad \frac{\forall x \neg \forall y (Pxy \rightarrow Qxy)}{\neg \forall y (Pay \rightarrow Qay)} \forall \text{Elim}}{\neg \forall y \neg Pay} \neg \text{Intro} \\ \frac{\neg \forall y \neg Pay}{\forall x \neg \forall y \neg Pxy} \forall \text{Intro} \end{array}$$

To prove $\forall y (Pay \rightarrow Qay)$ we need to apply \forall Intro, meaning we need to show that the statement is true for any arbitrary constant which could replace y . We cannot choose a as our arbitrary constant, because a appears in $\forall y (Pay \rightarrow Qay)$. Instead we will choose b : if we can derive $Pab \rightarrow Qab$ without b appearing in any undischarged assumptions, we can apply \forall Intro and derive $\forall y (Pay \rightarrow Qay)$:

$$\begin{array}{c} \forall x \neg \forall y (Pxy \rightarrow Qxy), [\forall y \neg Pay] \\ \vdots \\ \frac{\frac{Pab \rightarrow Qab}{\forall y (Pay \rightarrow Qay)} \forall \text{Intro} \quad \frac{\forall x \neg \forall y (Pxy \rightarrow Qxy)}{\neg \forall y (Pay \rightarrow Qay)} \forall \text{Elim}}{\neg \forall y \neg Pay} \neg \text{Intro} \\ \frac{\neg \forall y \neg Pay}{\forall x \neg \forall y \neg Pxy} \forall \text{Intro} \end{array}$$

Proving $Pab \rightarrow Qab$ is a simple case of assuming Pab and proving Qab . Note that Pab contains b , but we plan to apply \rightarrow Intro and discharge it before we reach the \forall Intro step where we go from arbitrary b to universal y .

$$\begin{array}{c} \forall x \neg \forall y (Pxy \rightarrow Qxy), [\forall y \neg Pay], [Pab] \\ \vdots \\ \frac{\frac{Qab}{Pab \rightarrow Qab} \rightarrow \text{Intro}}{\forall y (Pay \rightarrow Qay)} \forall \text{Intro} \quad \frac{\forall x \neg \forall y (Pxy \rightarrow Qxy)}{\neg \forall y (Pay \rightarrow Qay)} \forall \text{Elim}}{\neg \forall y \neg Pay} \neg \text{Intro} \\ \frac{\neg \forall y \neg Pay}{\forall x \neg \forall y \neg Pxy} \forall \text{Intro} \end{array}$$

We don't have a direct way of proving Qab , but the two assumptions we've made do give us a contradiction. From $\forall y \neg Pay$ we can derive $\neg Pab$, which contradicts our assumption of Pab . From this contradiction we can apply \neg Elim and derive Qab . It turns out that in this part of the proof we don't need to use our premise $\forall x \neg \forall y (Pxy \rightarrow Qxy)$ again.

$$\frac{\frac{\frac{[Pab] \quad \frac{[\forall y \neg Pay]}{\neg Pab} \forall\text{Elim}}{\neg Pab} \neg\text{Elim}}{Qab} \rightarrow\text{Intro}}{Pab \rightarrow Qab} \forall\text{Intro} \quad \frac{\forall x \neg \forall y (Pxy \rightarrow Qxy)}{\neg \forall y (Pay \rightarrow Qay)} \forall\text{Elim}}{\frac{\forall y (Pay \rightarrow Qay)}{\neg \forall y \neg Pay} \neg\text{Intro}} \forall\text{Intro}}{\frac{\neg \forall y \neg Pay}{\forall x \neg \forall y \neg Pxy} \forall\text{Intro}} \neg\text{Intro}$$

This gives us a complete proof, but it's worth verifying at this stage that both of our applications of \forall Intro are allowed.

The first time we apply \forall Intro, we move from $Pab \rightarrow Qab$ to $\forall y (Pay \rightarrow Qay)$. b doesn't appear in $\forall y (Pay \rightarrow Qay)$ or any undischarged assumptions in the proof of $\forall y (Pay \rightarrow Qay)$; Pab has already been discharged by this point.

The second time we apply \forall Intro, we move from $\neg \forall y \neg Pay$ to $\forall x \neg \forall y \neg Pxy$. a doesn't appear in $\forall x \neg \forall y \neg Pxy$ or any undischarged assumptions in the proof of $\forall x \neg \forall y \neg Pxy$; both Pab and $\forall y \neg Pay$ have been discharged by this point.

3.3 Existential quantifier

We can use the introduction and elimination rules for the existential quantifier to construct a proof of the following:

$$\exists x (Px \wedge Qx), \neg \exists x (Qx \wedge Rx) \vdash \exists x (Px \wedge \neg Rx)$$

The first thing to take note of is our existential premise $\exists x (Px \wedge Qx)$. In order to make use of it we need to apply \exists Elim at the end of the proof, discharging assumptions where x is instantiated with an arbitrary constant. We'll choose a as our arbitrary constant; we can use it because it doesn't appear in our premises ($\exists x (Px \wedge Qx)$ and $\neg \exists x (Qx \wedge Rx)$) or in the conclusion ($\exists x (Px \wedge \neg Rx)$). We'll also make sure that if a appears in any other assumptions, the assumptions are discharged by the time we apply \exists Elim.

This means our proof will take the following form:

$$\frac{\begin{array}{c} [Pa \wedge Qa], \neg \exists x (Qx \wedge Rx) \\ \vdots \\ \exists x (Px \wedge Qx) \quad \exists x (Px \wedge \neg Rx) \end{array}}{\exists x (Px \wedge \neg Rx)} \exists\text{Elim}$$

Our conclusion is also an existential statement, so we can prove it by applying \exists Intro. There is an infinite number of different statements which we could derive $\exists x(Px \wedge \neg Rx)$ from ($Pb \wedge \neg Rb$ and $Pc_{169} \wedge \neg Rc_{169}$ are two examples) but a is the only constant we have any assumptions about, so it seems likely that we will derive $\exists x(Px \wedge \neg Rx)$ from $Pa \wedge \neg Ra$:

$$\frac{\begin{array}{c} [Pa \wedge Qa], \neg \exists x(Qx \wedge Rx) \\ \vdots \\ \exists x(Px \wedge Qx) \end{array} \quad \frac{Pa \wedge \neg Ra}{\exists x(Px \wedge \neg Rx)} \exists \text{Intro}}{\exists x(Px \wedge \neg Rx)} \exists \text{Elim}$$

This is a conjunction, so using our premise and our assumption we need to provide a proof of Pa and a proof of $\neg Ra$:

$$\frac{\begin{array}{c} [Pa \wedge Qa], \quad [Pa \wedge Qa], \\ \neg \exists x(Qx \wedge Rx) \quad \neg \exists x(Qx \wedge Rx) \\ \vdots \quad \vdots \\ Pa \quad \neg Ra \end{array} \quad \wedge \text{Intro}}{\frac{\exists x(Px \wedge Qx) \quad \frac{Pa \wedge \neg Ra}{\exists x(Px \wedge \neg Rx)} \exists \text{Intro}}{\exists x(Px \wedge \neg Rx)} \exists \text{Elim}}$$

On the left-hand side, Pa is very easy to prove: it can be derived from our assumption of $Pa \wedge Qa$ by \wedge Elim:

$$\frac{\begin{array}{c} [Pa \wedge Qa], \\ \neg \exists x(Qx \wedge Rx) \\ \vdots \\ \frac{[Pa \wedge Qa]}{Pa} \wedge \text{Elim} \quad \neg Ra \end{array} \quad \wedge \text{Intro}}{\frac{\exists x(Px \wedge Qx) \quad \frac{Pa \wedge \neg Ra}{\exists x(Px \wedge \neg Rx)} \exists \text{Intro}}{\exists x(Px \wedge \neg Rx)} \exists \text{Elim}}$$

On the right-hand side, $\neg Ra$ is a negated statement. This means we can prove it by assuming Ra and deriving a contradiction.

Our other premise $\neg \exists x(Qx \wedge Rx)$ is a negated statement, so if we can prove $\exists x(Qx \wedge Rx)$ we have the contradiction we need:

$$\begin{array}{c}
[Pa \wedge Qa], [Ra], \\
\neg \exists x(Qx \wedge Rx) \\
\vdots \\
\frac{\frac{[Pa \wedge Qa]}{Pa} \wedge E \quad \frac{\exists x(Qx \wedge Rx) \quad \neg \exists x(Qx \wedge Rx)}{\neg Ra} \neg I}{Pa \wedge \neg Ra} \wedge I \\
\frac{\exists x(Px \wedge Qx) \quad \frac{\frac{Pa \wedge \neg Ra}{\exists x(Px \wedge \neg Rx)} \exists I}{\exists x(Px \wedge \neg Rx)} \exists E}{\exists x(Px \wedge \neg Rx)} \exists E
\end{array}$$

$\exists x(Qx \wedge Rx)$ can be derived from $Qa \wedge Ra$ by \exists Intro:

$$\begin{array}{c}
[Pa \wedge Qa], [Ra], \\
\neg \exists x(Qx \wedge Rx) \\
\vdots \\
\frac{\frac{[Pa \wedge Qa]}{Pa} \wedge E \quad \frac{\frac{Qa \wedge Ra}{\exists x(Qx \wedge Rx)} \exists I \quad \neg \exists x(Qx \wedge Rx)}{\neg Ra} \neg I}{Pa \wedge \neg Ra} \wedge I \\
\frac{\exists x(Px \wedge Qx) \quad \frac{\frac{Pa \wedge \neg Ra}{\exists x(Px \wedge \neg Rx)} \exists I}{\exists x(Px \wedge \neg Rx)} \exists E}{\exists x(Px \wedge \neg Rx)} \exists E
\end{array}$$

$Qa \wedge Ra$ is a conjunction, so we need to provide a proof of Qa and a proof of Ra . It turns out we don't need to use our premise $\neg \exists x(Qx \wedge Rx)$ again. We can obtain Qa from our assumption of $Pa \wedge Qa$, and we have Ra because we have assumed it in order to derive $\neg Ra$:

$$\begin{array}{c}
\frac{\frac{[Pa \wedge Qa]}{Qa} \wedge E \quad [Ra]}{Qa \wedge Ra} \wedge I \\
\frac{\frac{[Pa \wedge Qa]}{Pa} \wedge E \quad \frac{\frac{Qa \wedge Ra}{\exists x(Qx \wedge Rx)} \exists I \quad \neg \exists x(Qx \wedge Rx)}{\neg Ra} \neg I}{Pa \wedge \neg Ra} \wedge I \\
\frac{\exists x(Px \wedge Qx) \quad \frac{\frac{Pa \wedge \neg Ra}{\exists x(Px \wedge \neg Rx)} \exists I}{\exists x(Px \wedge \neg Rx)} \exists E}{\exists x(Px \wedge \neg Rx)} \exists E
\end{array}$$

This gives us a complete proof. Our assumption of Ra is discharged when we apply \neg Intro and derive $\neg Ra$. This means that by the time we apply \exists Elim, the only undischarged assumption left in the proof which involves a is $Pa \wedge Qa$. This means that at the end of the proof we are allowed to apply \exists Elim and discharge our two assumptions of $Pa \wedge Qa$.

4 Practice problems

4.1 Core

Knowing the rules

These seven proofs cover all of the Natural Deduction rules, and can be used to diagnose how familiar you are with the rules themselves and the strategies which correspond to them.

Learning the rules by heart is dull, but the best way is through practice. The later sections of the pack should have enough problems for each connective to establish a familiarity with them. If the seven proofs below prove to be straightforward, the later sections also have more challenging problems for each connective.

$$1. (P \rightarrow P) \leftrightarrow Q \vdash Q \vee R \quad (3)$$

$$2. \forall x(Px \vee Px) \vdash \forall xPx \quad (3)$$

$$3. \exists xPx \vdash \exists x(Px \vee Qx) \quad (3)$$

$$4. \neg P_1 \vdash \neg((P_1 \wedge P_2) \wedge P_3) \quad (3)$$

$$5. \neg P \vee Q \vdash P \rightarrow Q \quad (3)$$

$$6. Qa, \neg Qb \vdash a = a \wedge \neg a = b \quad (4)$$

$$7. \vdash (P \wedge Q) \leftrightarrow (Q \wedge P) \quad (7)$$

Making substitutions

The usual strategy for a proof involving quantifiers is to use the elimination rules to turn quantified statements into statements involving constants, manipulate those statements using the connective rules, and then turn those statements back into quantified statements with the introduction rules.

Once you've cracked how the quantifier rules work (including the nasty \exists Elim), the challenge becomes knowing which constants to substitute. Sometimes you will need to use your premises multiple times, making different substitutions each time. The proofs below test this; many more can be found in the quantifier sections of the pack.

$$8. \forall x \forall y (Rxy \rightarrow Ryx) \vdash \forall x \forall y (Rxy \leftrightarrow Ryx) \quad (7)$$

$$9. \forall x \exists y Rxy \vdash \forall x \exists y \exists z (Rxy \wedge Ryz) \quad (8)$$

$$10. \forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz), \forall x Rxx \vdash \forall x \forall y (Rxy \rightarrow Ryx) \quad (9)$$

Indirect proofs

Sometimes you find that, no matter how hard you try, you can't obtain the proof you want. This might be because you need an indirect proof: you prove a sentence ϕ by assuming $\neg\phi$ and showing that it leads to a contradiction.

Indirect kinds of proofs have often appeared in past papers. Part of the challenge is spotting them in the first place. Having to derive a disjunction without having any disjunctive premises is often a hint ($\vdash P \vee \neg P$ is the classic example). Similarly, you're likely to need an indirect proof to derive an existential statement from premises with no existential quantifiers (such as $\neg\forall x Px \vdash \exists x \neg Px$). Sometimes they can be harder to spot, which means it can be a good idea just to try an indirect proof if nothing else seems to be working.

There is a kind of indirect proof which is especially common. You prove ϕ by constructing a proof of the following shape:

$$\begin{array}{c} [\psi]^1 \\ \vdots \\ \phi \quad \frac{[\neg\phi]^2}{\neg\psi} \text{-Intro}^1 \\ \vdots \\ \phi \quad \frac{[\neg\phi]^2}{\phi} \text{-Elim}^2 \end{array}$$

Instead of simply proving ϕ , you show that ϕ follows from ψ and then from $\neg\psi$. The tricky part now is knowing which ψ to assume: usually you should look for a ψ which ϕ very easily follows from, but often there are many ψ s which result in proofs which work.

The examples below are all indirect proofs, including at least one with the special shape above. Many more indirect proofs are located in later sections of the pack, not always in obvious places.

$$11. \neg P \rightarrow Q \vdash P \vee Q \tag{5}$$

$$12. \vdash ((Q \rightarrow R) \rightarrow Q) \rightarrow Q \tag{5}$$

$$13. \neg\forall x\forall y Rxy \vdash \exists x\neg\forall y Rxy \tag{7}$$

$$14. Pa, Qb \vdash \exists x(Px \wedge Qx) \vee \exists x\exists y\neg x = y \tag{9}$$

4.2 Conjunction

$$1. P, Q \vdash P \wedge Q \quad (1)$$

$$2. (P_1 \wedge P_2) \wedge P_3 \vdash P_2 \quad (2)$$

$$3. P \wedge Q \vdash Q \wedge P \quad (3)$$

$$4. Q \wedge P, R \vdash P \wedge (R \wedge Q) \quad (4)$$

$$5. P_1 \wedge P_2, (Q_1 \wedge Q_2) \wedge R \vdash (P_1 \wedge Q_2) \wedge R \quad (6)$$

$$6. P \wedge (Q \wedge R) \vdash (R \wedge P) \wedge Q \quad (7)$$

4.3 Implication

1. $\vdash P \rightarrow P$ (1)
2. $\vdash P \rightarrow (Q \rightarrow P)$ (2)
3. $P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R$ (3)
4. $\vdash P \rightarrow ((P \rightarrow Q) \rightarrow Q)$ (3)
5. $(P \rightarrow Q) \rightarrow (P \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$ (3)
6. $(P \rightarrow Q) \rightarrow P \vdash Q \rightarrow P$ (3)
7. $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$ (4)
8. $P \rightarrow (Q \rightarrow R), P \rightarrow Q \vdash P \rightarrow R$ (4)
9. $(P \rightarrow P) \rightarrow Q \vdash (Q \rightarrow R) \rightarrow R$ (4)
10. $\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ (6)

Mixed problems with conjunction

11. $P \wedge Q \vdash P \rightarrow Q$ (2)
12. $\vdash P \wedge Q \rightarrow P$ (2)
13. $P \rightarrow (Q \wedge R) \vdash P \rightarrow Q$ (3)
14. $((P \wedge Q) \rightarrow Q) \rightarrow (Q \rightarrow P) \vdash Q \rightarrow P$ (3)
15. $(P \wedge Q) \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$ (4)
16. $(P \rightarrow Q) \wedge (P \rightarrow R) \vdash P \rightarrow (Q \wedge R)$ (6)
17. $P \rightarrow (Q \wedge R) \vdash (P \rightarrow Q) \wedge (P \rightarrow R)$ (7)

Bonus challenge

Provide a Natural Deduction proof of the following which consists of no more than eight steps:

$$(((P_1 \wedge P_2) \wedge P_3) \wedge P_4) \wedge P_5 \vdash P_1 \wedge P_1$$

(Here, a ‘step’ is considered to be any application of any rule, so the number of steps is equivalent to the number of times a horizontal line is drawn.)

4.4 Disjunction

$$1. P \vee Q \vdash Q \vee P \quad (3)$$

$$2. P \vee Q \vdash P \vee (Q \vee R) \quad (4)$$

$$3. (P \vee Q) \vee R \vdash P \vee (Q \vee R) \quad (7)$$

$$4. (P \vee Q) \vee (R \vee P_1) \vdash (P \vee P_1) \vee (R \vee Q) \quad (11)$$

Mixed problems with conjunction

$$5. P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R) \quad (8)$$

$$6. (P \vee Q) \wedge (P \vee R) \vdash P \vee (Q \wedge R) \quad (8)$$

$$7. (P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R) \quad (9)$$

$$8. P \vee (Q \wedge R) \vdash (P \vee Q) \wedge (P \vee R) \quad (9)$$

Mixed problems with implication

$$9. (P \rightarrow Q) \vee Q \vdash P \rightarrow Q \quad (2)$$

$$10. P \vee Q \vdash (P \rightarrow Q) \rightarrow Q \quad (3)$$

$$11. (P \rightarrow Q) \rightarrow (P \rightarrow R) \vdash (P \vee R) \rightarrow (Q \rightarrow R) \quad (6)$$

$$12. (P \rightarrow Q) \vee (P \rightarrow R) \vdash P \rightarrow (Q \vee R) \quad (7)$$

Mixed problems with conjunction and implication

$$13. (P \rightarrow Q) \wedge (Q \rightarrow P) \vdash (P \vee Q) \rightarrow (P \wedge Q) \quad (8)$$

$$14. (P \vee Q) \rightarrow (P \wedge Q) \vdash (P \rightarrow Q) \wedge (Q \rightarrow P) \quad (9)$$

$$15. (Q \rightarrow R) \wedge (Q \vee P) \vdash (P \rightarrow Q) \rightarrow (R \wedge Q) \quad (10)$$

4.5 Biconditional

$$1. P \leftrightarrow Q \vdash Q \leftrightarrow P \quad (3)$$

$$2. P, (P \leftrightarrow Q) \leftrightarrow R \vdash Q \leftrightarrow R \quad (5)$$

$$3. \vdash (P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P) \quad (7)$$

Mixed problems

$$4. (P \vee Q) \leftrightarrow Q \vdash P \rightarrow Q \quad (3)$$

$$5. (P \wedge Q) \leftrightarrow P \vdash P \rightarrow Q \quad (3)$$

$$6. P \rightarrow Q \vdash (P \vee Q) \leftrightarrow Q \quad (4)$$

$$7. P \rightarrow Q \vdash (P \wedge Q) \leftrightarrow P \quad (4)$$

$$8. (P \rightarrow Q) \wedge (Q \rightarrow P) \vdash P \leftrightarrow Q \quad (5)$$

$$9. \vdash (P \wedge Q) \rightarrow ((P \rightarrow Q) \rightarrow P) \quad (5)$$

$$10. \vdash ((P \rightarrow Q) \leftrightarrow P) \rightarrow (P \leftrightarrow Q) \quad (6)$$

$$11. ((P \vee Q) \leftrightarrow Q) \leftrightarrow P \vdash P \leftrightarrow Q \quad (7)$$

$$12. P \rightarrow (Q \leftrightarrow R) \vdash (P \wedge Q) \leftrightarrow (P \wedge R) \quad (13)$$

$$13. \vdash (P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R)) \quad (18)$$

4.6 Negation

Negation introduction

1. $P \vdash \neg\neg P$ (1)
2. $\neg P \vdash \neg(P \wedge Q)$ (2)
3. $P \rightarrow \neg P \vdash \neg P$ (2)
4. $\neg(P \rightarrow Q) \vdash \neg Q$ (2)
5. $\neg(P \wedge Q) \vdash P \rightarrow \neg Q$ (3)
6. $P \rightarrow Q \vdash \neg Q \rightarrow \neg P$ (3)
7. $\vdash \neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ (4)
8. $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$ (5)
9. $\neg P \vee \neg Q \vdash \neg(P \wedge Q)$ (5)

Ex falso quodlibet

10. $\neg P \vdash P \rightarrow Q$ (2)
11. $P \wedge \neg P \vdash Q$ (3)
12. $P \vee Q \vdash \neg P \rightarrow Q$ (3)
13. $P \rightarrow Q, P \wedge \neg Q \vdash R$ (4)
14. $P \vee Q, P \leftrightarrow Q, \neg(P \wedge Q) \vdash R$ (6)

Indirect proofs

15. $\neg\neg P \vdash P$ (1)
16. $\vdash P \vee \neg P$ (4)
17. $\neg(\neg P \vee \neg Q) \vdash P \wedge Q$ (5)
18. $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$ (6)

Mixed problems

19. $\neg(P \rightarrow Q) \vdash P$ (3)
20. $(P \rightarrow Q) \rightarrow P \vdash P$ (4)
21. $P \leftrightarrow \neg\neg Q \vdash P \leftrightarrow Q$ (5)
22. $(P \rightarrow Q) \rightarrow Q \vdash \neg Q \rightarrow P$ (5)

$$23. \neg P \wedge \neg Q \vdash \neg(P \vee Q) \quad (6)$$

$$24. \vdash P \vee (P \rightarrow Q) \quad (6)$$

$$25. \vdash (P \rightarrow Q) \vee (Q \rightarrow R) \quad (7)$$

$$26. \neg P \rightarrow Q, R \vee \neg Q, P \rightarrow (Q_1 \vee Q_2), \neg R \wedge \neg Q_2 \vdash Q_1 \quad (9)$$

$$27. P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R) \quad (11)$$

$$28. \vdash \neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q) \quad (12)$$

Bonus challenge 1

First, provide a proof of the following without using \neg -Elim:

$$\neg\neg\neg P \vdash \neg P$$

Second, provide a proof of the following without using \neg -Intro:

$$P \vdash \neg\neg P$$

Bonus challenge 2

Provide two different proofs of the following:

$$\neg\neg P \wedge \neg\neg Q \vdash P \wedge Q$$

The first proof should consist only of five steps (five applications of Natural Deduction rules).

In the second proof, you may only discharge assumptions using \neg -Elim in the final step of the proof. In other words, you may make any number of applications of \neg -Elim which don't discharge assumptions, but an application of \neg -Elim may only discharge assumptions if no rules are applied below it.

4.7 Universal quantifier

Unary predicates

$$1. \forall xPx \vdash \forall yPy \quad (2)$$

$$2. \vdash \forall x(Px \rightarrow Px) \quad (2)$$

$$3. \forall x(Pa \rightarrow Qx) \vdash Pa \rightarrow \forall zQz \quad (4)$$

$$4. \forall xPx \wedge \forall yQy \vdash \forall z(Pz \wedge Qz) \quad (5)$$

$$5. \forall x(Px \rightarrow Qx) \vdash \forall y_1Py_1 \rightarrow \forall y_2Qy_2 \quad (5)$$

$$6. \forall z(Pz \wedge Qz) \vdash \forall yPy \wedge \forall yQy \quad (5)$$

$$7. \forall x(Px \rightarrow Qx), \forall x\neg Qx \vdash \forall x\neg Px \quad (5)$$

$$8. \forall x_1Px_1 \vee \forall x_2Qx_2 \vdash \forall x(Px \vee Qx) \quad (6)$$

$$9. \forall x\forall y(Px \rightarrow Qy) \vdash \forall x(Px \rightarrow \forall zQz) \quad (6)$$

$$10. \forall x(Px \rightarrow Qx), \forall x(Qx \rightarrow Rx) \vdash \forall x(Px \rightarrow Rx) \quad (6)$$

$$11. \forall x(Px \vee Qx), \neg\forall xPx \vdash \neg\forall x\neg Qx \quad (6)$$

$$12. \forall x(Px \wedge Qx) \vdash \forall x\forall y(Px \wedge Qy) \quad (7)$$

Binary predicates

$$13. \forall x\forall yRxy \vdash \forall xRxx \quad (3)$$

$$14. \forall x\neg\forall yRxy \vdash \neg\forall x\forall yRxy \quad (3)$$

$$15. \forall xRxx \vdash \forall x\neg\forall y\neg Rxy \quad (4)$$

$$16. \forall x\neg Rxx \vdash \neg\forall x\forall y(Rxy \vee Ryx) \quad (5)$$

$$17. \neg\forall x\neg\forall yRyx \vdash \forall x\neg\forall y\neg Rxy \quad (6)$$

$$18. \forall xRxx \vdash \forall x\forall y(Rxy \rightarrow \neg\forall z\neg(Rxz \wedge Rzy)) \quad (6)$$

$$19. \forall x\forall yRxy \vdash \forall x(Rxx \wedge \forall yRyx) \quad (7)$$

$$20. \forall x\forall yRxy \vdash \forall x\forall y(Rxy \wedge Ryx) \quad (7)$$

$$21. \forall x\forall y(Rxy \rightarrow Ryx), \forall x\forall y\neg(Rxy \wedge Ryx) \vdash \forall x\forall y\neg Rxy \quad (9)$$

$$22. \forall x\forall y(Qxy \rightarrow Qyx), \forall x\forall y(\neg Qxy \vee \neg Qyx) \vdash \forall x\forall y\neg Qxy \quad (9)$$

$$23. \forall x\neg\forall y\neg Rxy \vdash \forall x\neg\forall y\forall z\neg(Rxy \wedge Ryz) \quad (11)$$

$$24. \forall x\forall y(Rxy \rightarrow Ryx), \forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz), \forall x\neg\forall y\neg Rxy \vdash \forall xRxx \quad (13)$$

$$25. \forall x \neg Rxx, \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \\ \vdash \forall x \forall y \forall z \neg ((Rxy \wedge Ryz) \wedge Rzx) \quad (16)$$

$$26. \forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz), \forall x Rxx \\ \vdash \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \quad (17)$$

4.8 Existential quantifier

Unary predicates

$$1. \exists xPx \vdash \exists yPy \quad (2)$$

$$2. \neg\exists xPx \vdash \exists x\neg Px \quad (3)$$

$$3. \exists x_1(Pa \rightarrow Qx_1) \vdash Pa \rightarrow \exists x_2Qx_2 \quad (4)$$

$$4. \exists x(Px \wedge Qx) \vdash \exists yPy \wedge \exists zQz \quad (6)$$

$$5. \exists x(Px \vee Qx) \vdash \exists yPy \vee \exists zQz \quad (6)$$

$$6. \exists xPx \vee \exists yQy \vdash \exists z(Pz \vee Qz) \quad (7)$$

$$7. Pa \rightarrow \exists xQx \vdash \exists x(Pa \rightarrow Qx) \quad (9)$$

Binary predicates

$$8. \vdash \exists x\exists y(Rxy \rightarrow Ryx) \quad (3)$$

$$9. \exists x\exists yRxy \vdash \exists x\exists yRyx \quad (4)$$

$$10. \exists xRxx \vdash \exists x\exists y(Rxy \wedge Ryx) \quad (4)$$

$$11. \neg\exists x\exists yRxy \vdash \neg\exists yRyy \quad (4)$$

$$12. \vdash \neg\exists x\exists y(Rxy \wedge \neg Rxy) \quad (6)$$

$$13. \vdash \exists xRxx \vee \exists x(Rxx \rightarrow \neg\exists yRyx) \quad (8)$$

$$14. Rab \wedge Rbc, \neg Qa, Qc \vdash \exists x\exists y((\neg Qx \wedge Qy) \wedge Rxy) \quad (12)$$

Ternary predicates

$$15. \exists x\exists y\neg\exists z\neg Pxyz \vdash \neg\exists x\neg\exists y\exists zPxyz \quad (8)$$

Mixed quantifier problems

$$16. \neg\exists xPx \vdash \forall x\neg Px \quad (3)$$

$$17. \exists x\neg Px \vdash \neg\forall xPx \quad (3)$$

$$18. \neg\forall xPx \vdash \exists x\neg Px \quad (4)$$

$$19. \forall x\neg Px \vdash \neg\exists xPx \quad (4)$$

$$20. \forall x(\exists yPy \rightarrow Qx) \vdash \forall x\exists y(Py \rightarrow Qx) \quad (6)$$

$$21. \forall x\neg\forall y(Pxy \rightarrow Qxy) \vdash \forall x\exists yPxy \quad (7)$$

$$22. \forall x(Pxx \vee \forall yQxy) \vdash \forall x(\exists yPxy \vee Qxx) \quad (7)$$

$$23. \exists x(Pxx \wedge \forall yQxy) \vdash \exists x(\exists yPxy \wedge Qxx) \quad (7)$$

$$24. \vdash \forall x\exists yRxy \vee \neg\forall xRxx \quad (7)$$

$$25. \forall x\exists yRxy \rightarrow \neg\exists xRxx, \exists x\forall yRyx \vdash \forall x\neg Rxx \quad (8)$$

$$26. \forall x(Px \rightarrow \exists yRyx) \vdash \exists zRza \vee \neg\forall xPx \quad (10)$$

$$27. \forall x\forall y\forall z(Rxy \vee Rzy \vee Rzx) \vdash \exists x\exists y\forall z(Rzx \vee Rzy) \quad (17)$$

Bonus challenge

Construct a Natural Deduction proof with the following two features:

- The proof consists of a single application of \exists Elim, and no applications of any other rules.
- By the end of the proof, one assumption is discharged and one is left undischarged.

4.9 Identity

$$1. \vdash \exists x \exists y x = y \quad (3)$$

$$2. a = b, \neg(b = b \wedge b = c) \vdash \neg a = c \quad (4)$$

$$3. \vdash a = b \leftrightarrow \forall x(x = a \rightarrow x = b) \quad (7)$$

$$4. \vdash \exists x \forall y x = y \leftrightarrow \forall x \forall y x = y \quad (9)$$

Unary predicates

$$5. Pa, \neg Pb \vdash \neg a = b \quad (2)$$

$$6. Pb \wedge Qb, \forall x(Px \rightarrow x = a) \vdash Qa \quad (5)$$

$$7. \exists x \forall y(Py \leftrightarrow x = y) \vdash \exists x \forall y(Py \rightarrow x = y) \quad (6)$$

$$8. \forall x(x = a \vee x = b), \exists x Px \vdash \neg Pa \rightarrow Pb \quad (7)$$

$$9. \exists x(Px \wedge Qx \wedge Rax), Pb \wedge Qb, \forall x \forall y((Px \wedge Qx) \wedge (Py \wedge Qy) \rightarrow x = y) \vdash Rab \quad (8)$$

$$10. \forall x \forall y(Px \wedge x = y \rightarrow \neg Qy) \vdash \forall z \neg(Pz \wedge Qz) \quad (9)$$

Binary predicates

$$11. \forall x \forall y(Rxy \leftrightarrow x = y) \vdash \forall x Rxx \quad (5)$$

$$12. \forall x \neg Rxx, Rab \vdash \exists x \exists y \neg x = y \quad (5)$$

$$13. \exists x \exists y Rxy, \exists x \forall y x = y \vdash \forall x \forall y Rxy \quad (9)$$

$$14. \vdash \forall x Pxx \leftrightarrow \forall x \forall y(\neg Pxy \rightarrow \neg x = y) \quad (13)$$

$$15. \forall x \exists y(Rxy \wedge Py), \forall x \neg Rxx \vdash \neg \forall x \forall y(Px \rightarrow (Py \rightarrow x = y)) \quad (14)$$

4.10 Additional challenges

Admissible rules

In this question we consider alternative rules which could be added to the system of Natural Deduction.

We say a rule is ‘admissible’ if we can add it to the system of Natural Deduction without changing which conclusions we can derive from which premises. In other words, a rule is *not* admissible if there is a set of sentences Γ and a sentence ϕ such that ϕ is provable from Γ with the rule, but ϕ is not provable from Γ in the original unaugmented system of Natural Deduction.

Which of the rules below are admissible? In each case, justify your answer: provide a proof of ϕ from Γ which would not be possible in the unaugmented system of Natural Deduction; or show that any proof of ϕ from Γ using the new rule can be rewritten using only the original Natural Deduction rules.

1.
$$\frac{\begin{array}{c} \vdots \\ \phi \wedge \psi \end{array} \quad \begin{array}{c} [\phi] \\ \vdots \\ \chi \end{array}}{\chi} \star 1$$
2.
$$\frac{\begin{array}{c} \vdots \\ \neg \phi \end{array} \quad \begin{array}{c} \vdots \\ \phi \rightarrow \psi \end{array}}{\neg \psi} \star 2$$

$$[\psi \rightarrow \phi]$$
3.
$$\frac{\begin{array}{c} \vdots \\ \phi \end{array}}{\phi} \star 3$$
4.
$$\frac{\begin{array}{c} [\neg \phi] \\ \vdots \\ \psi \end{array}}{\phi \vee \psi} \star 4$$

$$[\phi \rightarrow \psi]$$
5.
$$\frac{\begin{array}{c} \vdots \\ \phi \end{array}}{\phi} \star 5$$

Contraposition

Construct proofs of the following without using the \neg -Intro rule:

$$1. \neg(Q \rightarrow P) \vdash \neg P \quad (3)$$

$$2. \vdash \neg(P \wedge \neg P) \quad (5)$$

Show that:

3. Any application of \neg -Intro can be replaced by a subproof without \neg -Intro

Consider the rule (called contraposition) with the following representation:

$$\frac{\begin{array}{c} [\neg\psi] \\ \vdots \\ \neg\phi \end{array}}{\phi \rightarrow \psi} C$$

In the remaining questions of this section, we will explore how powerful the contraposition rule is.

Write $\Gamma \vdash_C \phi$ if there is a proof of ϕ from Γ in an alternate Natural Deduction system where the rule above can be used, and all original Natural Deduction rules except \rightarrow -Intro, \neg -Intro and \neg -Elim can be used.

We continue to write $\Gamma \vdash \phi$ if there is a proof of ϕ from Γ in the original system of Natural Deduction (i.e. with \neg -Intro, \neg -Elim and \rightarrow -Intro but without contraposition).

Show that:

$$4. \text{ If } \Gamma \vdash_C \phi \text{ then } \Gamma \vdash \phi$$

$$5. \neg\neg P \vdash_C \neg P \rightarrow \neg(Q \rightarrow Q) \quad (1)$$

$$6. \neg P \rightarrow \neg(Q \rightarrow Q) \vdash_C (Q \rightarrow Q) \rightarrow P \quad (2)$$

$$7. (Q \rightarrow Q) \rightarrow P \vdash_C P \quad (2)$$

$$8. \neg\neg P \vdash_C P \quad (4)$$

9. An application of \neg -Elim can be replaced by contraposition and \rightarrow -Elim

10. An application of \rightarrow -Intro can be replaced by contraposition and \neg -Intro

Using your answers to questions 3, 9 and 10, or otherwise, show that:

$$11. \text{ If } \Gamma \vdash \phi \text{ then } \Gamma \vdash_C \phi$$

5 Solutions

5.1 Core

Core 1

$$\frac{\frac{[P]}{P \rightarrow P} \rightarrow\text{Intro} \quad (P \rightarrow P) \leftrightarrow Q}{\frac{Q}{Q \vee R} \vee\text{Intro}} \leftrightarrow\text{Elim}$$

Our conclusion $Q \vee R$ is a disjunction, so we can either prove it by proving Q or by proving R . We're probably not going to be able to prove R since it doesn't appear in our premise $(P \rightarrow P) \leftrightarrow Q$, so instead we'll try and prove Q .

Our premise $(P \rightarrow P) \leftrightarrow Q$ is a biconditional, which means that if we can prove $P \rightarrow P$ we can use $\leftrightarrow\text{Elim}$ to derive Q . By assuming P , and then using $\rightarrow\text{Intro}$ to discharge this assumption of P , we can provide this proof of $P \rightarrow P$.

Core 2

$$\frac{\frac{\forall x(Px \vee Px)}{Pa \vee Pa} \vee\text{Elim} \quad [Pa] \quad [Pa]}{\frac{Pa}{\forall x Px} \vee\text{Intro}} \vee\text{Elim}$$

Our conclusion $\forall x Px$ is a universal statement. We can derive it from Pa using $\forall\text{Intro}$, as long as the constant a appears in no undischarged assumptions by the time we apply $\forall\text{Intro}$.

Our premise $\forall x(Px \vee Px)$ lets us derive $Pa \vee Pa$, a disjunction. Deriving Pa from $Pa \vee Pa$ requires a slightly bizarre use of $\vee\text{Elim}$: we make two assumptions of Pa and then immediately discharge them with $\vee\text{Elim}$ to derive Pa in one step. Because these assumptions of Pa have been discharged, we are free to apply $\forall\text{Intro}$ and derive the conclusion $\forall x Px$.

Core 3

$$\frac{\frac{\frac{[Pa]}{Pa \vee Qa} \vee\text{Intro}}{\exists x Px \quad \exists x(Px \vee Qx)} \exists\text{Intro}}{\exists x(Px \vee Qx)} \exists\text{Elim}$$

Our premise $\exists x Px$ is an existential statement, which means we need to use the dreaded $\exists\text{Elim}$ rule. We will use this rule at the very end of the proof to discharge an assumption of Pa .

From this assumption of Pa we can use $\vee\text{Intro}$ to derive $Pa \vee Qa$, and then $\exists\text{Intro}$ to derive the conclusion $\exists x(Px \vee Qx)$.

Our use of $\exists\text{Elim}$ at the end of the proof is allowed because the constant a doesn't appear in our premise $\exists x Px$, our conclusion $\exists x(Px \vee Qx)$ or any other assumptions we made when proving $\exists x(Px \vee Qx)$.

Core 4

$$\frac{\frac{\frac{[(P_1 \wedge P_2) \wedge P_3]}{P_1 \wedge P_2} \wedge\text{Elim}}{P_1} \wedge\text{Elim} \quad \neg P_1}{\neg((P_1 \wedge P_2) \wedge P_3)} \neg\text{Intro}$$

Our conclusion $\neg((P_1 \wedge P_2) \wedge P_3)$ is a negation, so we prove it by assuming $(P_1 \wedge P_2) \wedge P_3$ and deriving a contradiction. Our premise $\neg P_1$ is a negated statement, so we have a contradiction if we have a proof of P_1 .

We can prove P_1 from our assumption of $(P_1 \wedge P_2) \wedge P_3$ by using $\wedge\text{Elim}$ twice. Then with our premise $\neg P_1$ we can apply $\neg\text{Intro}$, discharge our assumption of $(P_1 \wedge P_2) \wedge P_3$ and derive $\neg((P_1 \wedge P_2) \wedge P_3)$.

Core 5

$$\frac{\frac{\neg P \vee Q \quad \frac{\frac{[P]^2 \quad [\neg P]^1}{Q} \neg\text{Elim}}{Q} \vee\text{Elim}^1}{P \rightarrow Q} \rightarrow\text{Intro}^2$$

Our conclusion $P \rightarrow Q$ is an implication, so we prove it by assuming P and deriving Q from it. Our premise $\neg P \vee Q$ is a disjunction, so we can use $\vee\text{Elim}$ to split the proof into a case where we can assume $\neg P$ and a case where we can assume Q .

The right-hand case is easy. We want to prove Q and we have an assumption of Q , so we don't need to do anything else. In the left-hand case, we can use our assumptions of P and $\neg P$ to derive Q by $\neg\text{Elim}$. Now we have established Q is true in both cases, we can use $\rightarrow\text{Intro}$ to derive $P \rightarrow Q$.

Core 6

$$\frac{\frac{Qa \quad [a = b]}{Qb} \text{=Elim} \quad \neg Qb}{[a = a]^= \quad \neg a = b} \text{-Intro} \quad \frac{}{a = a \wedge \neg a = b} \wedge\text{Intro}$$

Our conclusion $a = a \wedge \neg a = b$ is a conjunction, so we need to provide two proofs: a proof of $a = a$ and a proof of $\neg a = b$.

The left-hand proof is easy. We can assume $a = a$ and immediately discharge it using =Intro.

The right-hand proof is a proof of a negated statement, so we assume $a = b$ and try to derive a contradiction from it. Since one of our premises $\neg Pb$ is a negated statement, we have the contradiction we need if we can prove Pb . We prove Pb by using =Elim together with our other premise Pa and our assumption of $a = b$.

Core 7

$$\frac{\frac{[P \wedge Q]}{Q} \wedge\text{Elim} \quad \frac{[P \wedge Q]}{P} \wedge\text{Elim}}{Q \wedge P} \wedge\text{Intro} \quad \frac{\frac{[Q \wedge P]}{P} \wedge\text{Elim} \quad \frac{[Q \wedge P]}{Q} \wedge\text{Elim}}{P \wedge Q} \wedge\text{Intro}}{(P \wedge Q) \leftrightarrow (Q \wedge P)} \leftrightarrow\text{Intro}$$

The conclusion $(P \wedge Q) \leftrightarrow (Q \wedge P)$ is a biconditional, so we prove it by providing a proof of $Q \wedge P$ from assumptions of $P \wedge Q$ and a proof of $P \wedge Q$ from assumptions of $Q \wedge P$.

Both of the sides of the proof work similarly. We use $\wedge\text{Elim}$ with our assumption to obtain P and Q , and then we use $\wedge\text{Intro}$ to derive $Q \wedge P$ on the left and $P \wedge Q$ on the right.

Core 8

$$\begin{array}{c}
\frac{\frac{\forall x\forall y(Rxy \rightarrow Ryx)}{\forall y(Ray \rightarrow Rya)} \forall E}{Rab \rightarrow Rba} \forall E \quad [Rab] \rightarrow E \quad \frac{\frac{\forall x\forall y(Rxy \rightarrow Ryx)}{\forall y(Rby \rightarrow Ryb)} \forall E}{Rba \rightarrow Rab} \forall E \quad [Rba] \rightarrow E}{\frac{Rba}{Rab} \leftrightarrow I} \leftrightarrow I \\
\frac{Rab \leftrightarrow Rba}{\forall y(Ray \leftrightarrow Rya)} \forall I \\
\frac{\forall y(Ray \leftrightarrow Rya)}{\forall x\forall y(Rxy \leftrightarrow Ryx)} \forall I
\end{array}$$

This is a past paper question from 2015. Our conclusion $\forall x\forall y(Rxy \leftrightarrow Ryx)$ is a universal statement, which we can derive from $\forall y(Ray \leftrightarrow Rya)$. This in turn we can derive from $Rab \leftrightarrow Rba$. In order to prove this, we need to provide a proof of Rba from Rab and of Rab from Rba .

On the left-hand side we assume Rab and derive $Rab \rightarrow Rba$ from our premise $\forall x\forall y(Rxy \rightarrow Ryx)$, which gives us Rba . The right-hand side is similar: we assume Rba and derive $Rba \rightarrow Rab$ from the premise, which gives us Rab . Both of these assumptions are discharged when we apply \leftrightarrow Intro, so we are free to apply \forall Intro.

Core 9

$$\begin{array}{c}
\frac{\frac{\forall x\exists yRxy}{\exists yRay} \forall Elim \quad \frac{\frac{\forall x\exists yRxy}{\exists yRby} \forall Elim \quad \frac{\frac{[Rab]^2 \quad [Rbc]^1}{Rab \wedge Rbc} \wedge Intro}{\exists z(Rab \wedge Rbz)} \exists Intro}{\exists y\exists z(Ray \wedge Ryz)} \exists Intro}{\exists y\exists z(Ray \wedge Ryz)} \exists Elim^1}{\frac{\exists y\exists z(Ray \wedge Ryz)}{\forall x\exists y\exists z(Rxy \wedge Ryz)} \forall Intro} \exists Elim^2
\end{array}$$

This is adapted from a past paper question from 2012. Our conclusion $\forall x\exists y\exists z(Rxy \wedge Ryz)$ is a universal statement, so we can prove it by proving $\exists y\exists z(Ray \wedge Ryz)$ as long as a doesn't appear in any undischarged assumptions by the time we apply \forall Intro.

From our premise $\forall x\exists yRxy$ we can derive the existential statement $\exists yRay$, which lets us discharge an assumption of Rab by \exists Elim (as long as b doesn't appear in any other undischarged assumptions when we apply \exists Elim). However, Rab alone isn't enough to derive $\exists y\exists z(Ray \wedge Ryz)$. We need to use the premise a second time to derive $\exists yRby$, which lets us discharge an assumption of Rbc . With these assumptions of Rab and Rbc we can derive $\exists y\exists z(Ray \wedge Ryz)$.

We need to be careful about the order in we apply the \exists Elim steps. If we tried to discharge Rab before Rbc , we wouldn't be allowed to, because at that stage b still appears in an undischarged assumption. Instead we use $\exists yRay$ first to discharge Rbc , and then use $\exists yRay$ to discharge Rab .

Core 10

$$\begin{array}{c}
 \frac{\frac{[Rab] \quad \frac{\forall x Rxx}{Raa} \text{ } \forall\text{Elim}}{Rab \wedge Raa} \wedge\text{Intro} \quad \frac{\frac{\frac{\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz)}{\forall y \forall z ((Ray \wedge Raz) \rightarrow Ryz)} \text{ } \forall\text{Elim}}{\forall z ((Rab \wedge Raz) \rightarrow Rbz)} \text{ } \forall\text{Elim}}{(Rab \wedge Raa) \rightarrow Rba} \text{ } \forall\text{Elim}}{\rightarrow\text{Elim}} \\
 \frac{\frac{Rba}{Rab \rightarrow Rba} \text{ } \rightarrow\text{Intro}}{\forall y (Ray \rightarrow Rya)} \text{ } \forall\text{Intro}}{\forall x \forall y (Rxy \rightarrow Ryx)} \text{ } \forall\text{Intro}
 \end{array}$$

Our conclusion $\forall x \forall y (Rxy \rightarrow Ryx)$ features two universal quantifiers; we can prove it by proving $Rab \rightarrow Rba$ without a or b appearing in any undischarged assumptions. To do this we assume Rab and prove Rba .

We can't obtain Rba from Rab alone, but from one of our premises we can derive Raa , which allows us to derive $Rab \wedge Rba$. From our other premise we can obtain $(Rab \wedge Rba) \rightarrow Rba$, giving us the proof of Rba we need. Our assumption of Rab is discharged when we apply $\rightarrow\text{Intro}$, so we are free to apply $\forall\text{Intro}$ and derive our conclusion.

Core 11

$$\begin{array}{c}
 \frac{\frac{[P]^1}{P \vee Q} \text{ } \forall\text{Intro} \quad [\neg(P \vee Q)]^2}{\neg P} \text{ } \neg\text{Intro}^1 \quad \neg P \rightarrow Q}{\rightarrow\text{Elim}} \\
 \frac{\frac{Q}{P \vee Q} \text{ } \forall\text{Intro} \quad [\neg(P \vee Q)]^2}{P \vee Q} \text{ } \neg\text{Elim}^2
 \end{array}$$

We want to try to prove $P \vee Q$ but our premise $\neg P \rightarrow Q$ doesn't give us a direct proof of P or a direct proof of Q . We will need to assume $\neg(P \vee Q)$ and show that it leads to a contradiction.

We start by assuming P . From this, we can derive $P \vee Q$, which contradicts our assumption of $\neg(P \vee Q)$. Even though we have a contradiction, we can't immediately conclude $P \vee Q$, because our assumption of P is undischarged. Instead, we apply $\neg\text{Intro}$ to discharge P and prove $\neg P$.

This is where our premise comes in. We have proved $\neg P$, and our premise $\neg P \rightarrow Q$ allows us to derive Q . From this, we can derive $P \vee Q$ again, so assuming $\neg(P \vee Q)$ again leads to a contradiction. From this contradiction we can discharge our assumptions of $\neg(P \vee Q)$ and derive $P \vee Q$.

Core 12

$$\frac{\frac{\frac{[Q]^1 \quad [-Q]^2}{R} \text{-Elim}}{Q \rightarrow R} \text{-Intro}^1 \quad \frac{[(Q \rightarrow R) \rightarrow Q]^3}{Q} \text{-Elim} \quad [-Q]^2 \text{-Elim}^2}{\frac{Q}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \text{-Intro}^3}$$

There are many ways of carrying out this proof; shown above is one of the shortest possible methods. To prove $((Q \rightarrow R) \rightarrow Q) \rightarrow Q$, an implication, we assume $((Q \rightarrow R) \rightarrow Q)$ and provide a proof of Q . We can't provide a direct proof of Q , so instead we assume $\neg Q$ and show that it leads to a contradiction.

Since $\neg Q$ is itself a negated statement, we can show that it leads to a contradiction if we can show it leads to Q . We have assumed $(Q \rightarrow R) \rightarrow Q$, which is an implication with Q as its consequent. This means that if we can derive $Q \rightarrow R$ from $\neg Q$, we will be able to obtain Q and hence a contradiction.

Deriving $Q \rightarrow R$ from $\neg Q$ isn't too tricky. Since $Q \rightarrow R$ is an implication, we can assume Q , and Q and $\neg Q$ together give us R by \neg -Elim.

We could have tried to prove $((Q \rightarrow R) \rightarrow Q) \rightarrow Q$ by assuming $\neg(((Q \rightarrow R) \rightarrow Q) \rightarrow Q)$ and deriving a contradiction from that. Doing this will result in a longer proof, but it is still possible. Two examples below illustrate this approach.

$$\frac{[Q]^2 \quad \frac{\frac{[Q]^1}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow I \quad \frac{[¬(((Q \rightarrow R) \rightarrow Q) \rightarrow Q)]^4}{¬Q} \rightarrow E}{¬Q} \rightarrow E \quad \neg\text{Intro}^1}{\frac{R}{Q \rightarrow R} \rightarrow I^2 \quad \frac{Q}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow I^3 \quad \frac{[¬(((Q \rightarrow R) \rightarrow Q) \rightarrow Q)]^4}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow E \quad \neg E^4}}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow E$$

$$\frac{[Q]^1 \quad \frac{[¬Q]^4}{R} \rightarrow E \quad \frac{[Q \rightarrow R]^3 \quad [¬(((Q \rightarrow R) \rightarrow Q) \rightarrow Q)]^2}{Q} \rightarrow E \quad \frac{Q}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow I^2 \quad \frac{[¬(((Q \rightarrow R) \rightarrow Q) \rightarrow Q)]^5}{¬(Q \rightarrow R)} \rightarrow I^3 \quad \neg E^4}{Q} \rightarrow I^1 \quad \neg E^4}{\frac{Q}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow I \quad \frac{[¬(((Q \rightarrow R) \rightarrow Q) \rightarrow Q)]^5}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow E^5}}{((Q \rightarrow R) \rightarrow Q) \rightarrow Q} \rightarrow E^5$$

Core 13

$$\begin{array}{c}
 \frac{[\forall yRay]^1}{Rab} \forall\text{Elim} \quad [\neg Rab]^2}{\frac{\neg\forall yRay}{\exists x\neg\forall yRxy} \exists\text{Intro}} \neg\text{Intro}^1 \quad [\neg\exists x\neg\forall yRxy]^3}{\frac{\frac{Rab}{\forall yRay} \forall\text{Intro}}{\forall x\forall yRxy} \forall\text{Intro}}{\exists x\neg\forall yRxy} \neg\text{Elim}^2} \neg\text{Elim}^3
 \end{array}$$

This is a very awkward proof. Our conclusion is $\exists x\neg\forall yRxy$, but we cannot prove it directly by $\exists\text{Intro}$. Instead we assume $\neg\exists x\neg\forall yRxy$ and derive a contradiction from it. Since our premise $\neg\forall x\forall yRxy$ is a negated statement, we have a contradiction if we can derive $\forall x\forall yRxy$ from $\neg\exists x\neg\forall yRxy$.

We can derive $\forall x\forall yRxy$ from Rab , provided that neither a nor b appears in any undischarged assumptions by the time we've derived Rab . But we have no way of proving Rab directly either; instead we need to make another indirect proof, assuming $\neg Rab$ and deriving a contradiction from that.

Since $\neg\exists x\neg\forall yRxy$ is a negated statement, we have a contradiction if we can derive $\exists x\neg\forall yRxy$ from $\neg Rab$. This time we do have a way of proving $\exists x\neg\forall yRxy$ from $\neg Rab$ directly: it follows from $\neg\forall yRay$. This in turn is a negated statement, so we can derive it if we assume $\forall yRay$ and derive a contradiction from it. We do this by deriving Rab , which contradicts our assumption of $\neg Rab$.

Core 14

$$\begin{array}{c}
 \frac{Pa \quad \frac{Qb \quad [a = b]}{Qa} =\text{Elim}}{Pa \wedge Qa} \wedge\text{Intro} \\
 \frac{}{\exists x(Px \wedge Qx)} \exists\text{Intro} \\
 \frac{}{\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y} \vee\text{Intro} \quad [\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)] \\
 \frac{}{\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y} \neg\text{Intro} \\
 \frac{}{\neg a = b} \neg\text{Intro} \\
 \frac{}{\exists y \neg a = y} \exists\text{Intro} \\
 \frac{}{\exists x \exists y \neg x = y} \exists\text{Intro} \\
 \frac{}{\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y} \vee\text{Intro} \quad [\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)] \\
 \frac{}{\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y} \neg\text{Elim}
 \end{array}$$

Our conclusion $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$ is a disjunction and we have no disjunctive premises. This means we probably need to prove $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$ indirectly: we assume $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$ and show that it leads to a contradiction.

We start by assuming $a = b$, since this lets us easily prove Qa , $Pa \wedge Qa$ and $\exists x(Px \wedge Qx)$. From there we derive $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$, which contradicts our assumption of $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$. With this contradiction we use $\neg\text{Intro}$ to discharge $a = b$ and derive $\neg a = b$.

From $\neg a = b$ we can also obtain our conclusion. By applying $\exists\text{Intro}$ twice we can derive $\exists x \exists y \neg x = y$, and from there we can derive $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$. Finally we assume $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$ again and apply $\neg\text{Elim}$, discharging both assumptions of $\neg(\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y)$ and giving us the conclusion $\exists x(Px \wedge Qx) \vee \exists x \exists y \neg x = y$.

5.2 Conjunction

Conjunction 1

$$\frac{P \quad Q}{P \wedge Q} \wedge\text{Intro}$$

Conjunction 2

$$\frac{\frac{(P_1 \wedge P_2) \wedge P_3}{P_1 \wedge P_2} \wedge\text{Elim1}}{P_2} \wedge\text{Elim2}$$

Conjunction 3

$$\frac{\frac{P \wedge Q}{Q} \wedge\text{Elim2} \quad \frac{P \wedge Q}{P} \wedge\text{Elim1}}{Q \wedge P} \wedge\text{Intro}$$

Conjunction 4

$$\frac{\frac{Q \wedge P}{P} \wedge\text{Elim2} \quad \frac{R \quad \frac{Q \wedge P}{Q} \wedge\text{Elim1}}{R \wedge Q} \wedge\text{Intro}}{P \wedge (R \wedge Q)} \wedge\text{Intro}$$

Conjunction 5

$$\frac{\frac{\frac{P_1 \wedge P_2}{P_1} \wedge\text{Elim1} \quad \frac{\frac{(Q_1 \wedge Q_2) \wedge R}{Q_1 \wedge Q_2} \wedge\text{Elim1}}{Q_2} \wedge\text{Elim2}}{P_1 \wedge Q_1} \wedge\text{Intro} \quad \frac{(Q_1 \wedge Q_2) \wedge R}{R} \wedge\text{Elim2}}{(P_1 \wedge Q_2) \wedge R} \wedge\text{Intro}$$

Conjunction 6

$$\frac{\frac{\frac{P \wedge (Q \wedge R)}{Q \wedge R} \wedge\text{Elim2}}{R} \wedge\text{Elim2} \quad \frac{P \wedge (Q \wedge R)}{P} \wedge\text{Elim1} \quad \frac{P \wedge (Q \wedge R)}{Q \wedge R} \wedge\text{Elim2}}{\frac{R \wedge P}{Q} \wedge\text{Elim1}} \wedge\text{Intro} \quad \frac{(R \wedge P) \wedge Q}{Q} \wedge\text{Intro}$$

5.3 Implication

Implication 1

$$\frac{[P]}{P \rightarrow P} \rightarrow\text{Intro}$$

This proof relies on a special case of the $\rightarrow\text{Intro}$ rule: both ϕ and ψ are the same. That means when we apply the rule we discharge P and put P on both sides of the arrow.

Implication 2

$$\frac{\frac{[P]}{Q \rightarrow P} \rightarrow\text{Intro}}{P \rightarrow (Q \rightarrow P)} \rightarrow\text{Intro}$$

In the first step of this proof, we discharge all assumptions of Q , but don't actually discharge any assumptions. We can go straight from P to $Q \rightarrow P$. It's in the second step that we discharge our original assumption of P .

Implication 3

$$\frac{\frac{[P] \quad P \rightarrow Q}{Q} \rightarrow\text{Elim} \quad Q \rightarrow R}{\frac{R}{P \rightarrow R} \rightarrow\text{Intro}} \rightarrow\text{Elim}$$

Here we can freely assume P , and we want to try to get to R . This is nice and easy P takes us to Q because we have $P \rightarrow Q$ as a premise, and then Q takes us to R because we have $Q \rightarrow R$ as a premise. The final application of $\rightarrow\text{Intro}$ discharges our assumption of P .

Implication 4

$$\frac{\frac{\frac{[P]^2 \quad [P \rightarrow Q]^1}{Q} \rightarrow\text{Elim}}{(P \rightarrow Q) \rightarrow Q} \rightarrow\text{Intro}^1}{P \rightarrow ((P \rightarrow Q) \rightarrow Q)} \rightarrow\text{Intro}^2$$

In order to prove $P \rightarrow ((P \rightarrow Q) \rightarrow Q)$, we have to prove $(P \rightarrow Q) \rightarrow Q$, and we're allowed to freely assume P . In order to prove $(P \rightarrow Q) \rightarrow Q$, we have to prove Q , and we're allowed to freely assume $P \rightarrow Q$. With assumptions of both P and $P \rightarrow Q$ we can use $\rightarrow\text{Elim}$ to get Q . Then the two $\rightarrow\text{Intro}$ steps discharge the two assumptions of $P \rightarrow Q$ and P .

Implication 5

$$\frac{\frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro} \quad (P \rightarrow Q) \rightarrow (P \rightarrow R)}{\frac{P \rightarrow R}{Q \rightarrow (P \rightarrow R)} \rightarrow\text{Intro}} \rightarrow\text{Elim}$$

Our conclusion is $Q \rightarrow (P \rightarrow R)$, so we should try to prove $P \rightarrow R$ from assumptions of Q . In the first step of the proof, we apply $\rightarrow\text{Intro}$ without discharging anything: Q takes us straight to $P \rightarrow Q$. After that, our big premise gets us to $P \rightarrow R$, and our final application of $\rightarrow\text{Intro}$ discharges Q .

Implication 6

$$\frac{\frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro} \quad (P \rightarrow Q) \rightarrow P}{\frac{P}{Q \rightarrow P} \rightarrow\text{Intro}} \rightarrow\text{Elim}$$

Our conclusion is $Q \rightarrow P$, so we can assume Q and need to prove P . Our premise $(P \rightarrow Q) \rightarrow P$ is an implication with P as its consequent, so we know we can derive P if we can provide a proof of $P \rightarrow Q$. This follows from Q by $\rightarrow\text{Intro}$ (another time when we apply $\rightarrow\text{Intro}$ without discharging anything), so we can then apply $\rightarrow\text{Elim}$ to derive P . Finally we apply $\rightarrow\text{Intro}$ to discharge Q and derive $Q \rightarrow P$.

Implication 7

$$\frac{\frac{[P] \quad P \rightarrow (Q \rightarrow R)}{Q \rightarrow R} \rightarrow\text{Elim} \quad [Q]}{\frac{\frac{R}{P \rightarrow R} \rightarrow\text{Intro}}{Q \rightarrow (P \rightarrow R)} \rightarrow\text{Intro}} \rightarrow\text{Elim}$$

Here we can freely assume P and Q , and we need to get to R . Our premise gets us from the P we've assumed to $Q \rightarrow R$, and then the Q we've assumed takes us to R .

Implication 8

$$\frac{\frac{[P] \quad P \rightarrow (Q \rightarrow R)}{Q \rightarrow R} \rightarrow\text{Elim} \quad \frac{[P] \quad P \rightarrow Q}{Q} \rightarrow\text{Elim}}{\frac{R}{P \rightarrow R} \rightarrow\text{Intro}} \rightarrow\text{Elim}$$

In this proof the conclusion is quite simple. Having to prove $P \rightarrow R$ means we can only assume P and only need to get to R . But this time we have two premises, and the P we've assumed works with both the premises: it gives us both the $Q \rightarrow R$ on the left and the Q on the right that we need to get to R .

Implication 9

$$\frac{\frac{\frac{[P]}{P \rightarrow P} \rightarrow \text{Intro} \quad (P \rightarrow P) \rightarrow Q}{Q} \rightarrow \text{Elim} \quad [Q \rightarrow R]}{\frac{R}{(Q \rightarrow R) \rightarrow R} \rightarrow \text{Intro}} \rightarrow \text{Elim}$$

This is a past paper question from 2009. In order to prove $(Q \rightarrow R) \rightarrow R$, we can freely assume $Q \rightarrow R$ and need to derive R . Because we can freely assume $Q \rightarrow R$, we know that we can get R if we can somehow prove Q . Because our premise is $(P \rightarrow P) \rightarrow Q$, we know that we can get Q if we can somehow prove $P \rightarrow P$. $P \rightarrow P$ is easy to prove: we assume P , and then apply \rightarrow Intro to discharge the assumption of P and prove $P \rightarrow P$. This then gives us Q , which then gives us R .

Implication 10

$$\frac{\frac{\frac{[P]^1 \quad [P \rightarrow (Q \rightarrow R)]^3}{Q \rightarrow R} \rightarrow \text{Elim} \quad \frac{[P]^1 \quad [P \rightarrow Q]^2}{Q} \rightarrow \text{Elim}}{\frac{R}{P \rightarrow R} \rightarrow \text{Intro}^1} \rightarrow \text{Elim}}{\frac{(P \rightarrow Q) \rightarrow (P \rightarrow R)}{(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)} \rightarrow \text{Intro}^2} \rightarrow \text{Intro}^3$$

This proof looks nasty, but it turns out to be systematic. First of all we look at how the conclusion is composed: $P \rightarrow (Q \rightarrow R)$ is the antecedent (so we can freely assume that) and $(P \rightarrow Q) \rightarrow (P \rightarrow R)$ on the consequent, so that's what we need to prove.

$(P \rightarrow Q) \rightarrow (P \rightarrow R)$ is what we need to prove first. That means we can freely assume $P \rightarrow Q$, and we need to prove $P \rightarrow R$. To prove $P \rightarrow R$, we can freely assume P , and need to derive R .

So all this has allowed us to freely assume $P \rightarrow (Q \rightarrow R)$, $P \rightarrow Q$ and P . The P and $P \rightarrow Q$ together give us Q , which you can see on the right. The P and the $P \rightarrow (Q \rightarrow R)$ give us $Q \rightarrow R$, which you can see on the left. Together, the Q and the $Q \rightarrow R$ give us R . Then it's just a case of working backwards from there, building up the conclusion and systematically discharging all the assumptions we've made with three \rightarrow Intro steps.

Implication 11

$$\frac{\frac{P \wedge Q}{Q} \wedge \text{Elim}}{P \rightarrow Q} \rightarrow \text{Intro}$$

This is a simpler proof than it might look: $P \wedge Q$ takes us straight to Q , which then takes us to $P \rightarrow Q$ without us needing to discharge anything.

Implication 12

$$\frac{\frac{[P \wedge Q]}{P} \wedge\text{Elim}}{P \wedge Q \rightarrow P} \rightarrow\text{Intro}$$

Recall that, according to the bracketing conventions, $P \wedge Q \rightarrow P$ is an abbreviation of $(P \wedge Q) \rightarrow P$. This is an implication, meaning we can freely assume the antecedent $P \wedge Q$ and need to derive the consequent P . We can get from $P \wedge Q$ to P in one step using $\wedge\text{Elim}$. Then all we need to do is apply $\rightarrow\text{Intro}$ discharge the $P \wedge Q$ and derive $P \wedge Q \rightarrow P$.

Implication 13

$$\frac{\frac{[P] \quad P \rightarrow (Q \wedge R)}{Q \wedge R} \rightarrow\text{Elim}}{\frac{Q}{P \rightarrow Q} \wedge\text{Elim}} \rightarrow\text{Intro}$$

Here we can freely assume P and need to get to Q . Our assumption of P allows us to get at the $Q \wedge R$ in the premise, which then gives us Q .

Implication 14

$$\frac{\frac{\frac{[P \wedge Q]}{Q} \wedge\text{Elim}}{(P \wedge Q) \rightarrow Q} \rightarrow\text{Intro}}{Q \rightarrow P} \rightarrow\text{Elim}}{((P \wedge Q) \rightarrow Q) \rightarrow (Q \rightarrow P)} \rightarrow\text{Elim}$$

Although our conclusion $Q \rightarrow P$ is an implication, we won't prove it by assuming Q and deriving P from it. Instead, notice that our premise $((P \wedge Q) \rightarrow Q) \rightarrow (Q \rightarrow P)$ is an implication with $Q \rightarrow P$ as its consequent. This means that if we can prove $(P \wedge Q) \rightarrow Q$ we can derive $Q \rightarrow P$ directly by $\rightarrow\text{Elim}$. $(P \wedge Q) \rightarrow Q$ can be proved by assuming $P \wedge Q$ and deriving Q by $\wedge\text{Elim}$.

Implication 15

$$\frac{\frac{\frac{[P]^2 \quad [Q]^1}{P \wedge Q} \wedge\text{Intro} \quad (P \wedge Q) \rightarrow R}{R} \rightarrow\text{Elim}}{\frac{R}{Q \rightarrow R} \rightarrow\text{Intro}^1} \rightarrow\text{Intro}^2}{P \rightarrow (Q \rightarrow R)}$$

Because our conclusion $P \rightarrow (Q \rightarrow R)$ is an implication, we know that we need to prove $Q \rightarrow R$ from assumptions of P . In order to prove $Q \rightarrow R$, we need to prove R and can freely assume Q . Our assumptions of P and Q together give us $P \wedge Q$; that, combined with the premise, gives us R . Then we apply $\rightarrow\text{Intro}$ twice to discharge our assumptions of P and Q .

Implication 16

$$\frac{\frac{[P] \quad \frac{(P \rightarrow Q) \wedge (P \rightarrow R)}{P \rightarrow Q} \wedge\text{Elim}}{Q} \rightarrow\text{Elim} \quad \frac{[P] \quad \frac{(P \rightarrow Q) \wedge (P \rightarrow R)}{P \rightarrow R} \wedge\text{Elim}}{R} \rightarrow\text{Elim}}{\frac{Q \wedge R}{P \rightarrow (Q \wedge R)} \wedge\text{Intro} \rightarrow\text{Intro}}$$

The main connective in the conclusion $P \rightarrow (Q \wedge R)$ is an arrow. That means we can freely assume P and need to get to $Q \wedge R$. To get to $Q \wedge R$, we need two separate proofs: one of Q and one of R . Both of these work in a similar way. We split open the original premise $(P \rightarrow Q) \wedge (P \rightarrow R)$ to get conditionals $P \rightarrow Q$ and $P \rightarrow R$, and then we use our assumption of P to give us both Q and R .

Implication 17

$$\frac{\frac{[P] \quad \frac{P \rightarrow (Q \wedge R)}{Q \wedge R} \rightarrow\text{Elim}}{Q} \wedge\text{Elim} \rightarrow\text{Intro} \quad \frac{[P] \quad \frac{P \rightarrow (Q \wedge R)}{Q \wedge R} \rightarrow\text{Elim}}{R} \wedge\text{Elim} \rightarrow\text{Intro}}{\frac{P \rightarrow Q \quad P \rightarrow R}{(P \rightarrow Q) \wedge (P \rightarrow R)} \wedge\text{Intro}}$$

Because the conclusion $(P \rightarrow Q) \wedge (P \rightarrow R)$ is a conjunction, we know we need to do two proofs: one proof of what's on the left, and one proof of what's on the right. In each proof we can freely assume P (which we know we'll discharge each time we apply $\rightarrow\text{Intro}$), which works with the premise $(P \rightarrow (Q \wedge R))$ to give us $Q \wedge R$, and from there the Q and the R we need.

5.4 Disjunction

Disjunction 1

$$\frac{P \vee Q \quad \frac{[P]}{Q \vee P} \vee\text{Intro2} \quad \frac{[Q]}{Q \vee P} \vee\text{Intro1}}{Q \vee P} \vee\text{Elim}$$

Our premise $P \vee Q$ splits the proof up into two cases: one where P is true and one where Q is true. Both of these are cases of $Q \vee P$.

Disjunction 2

$$\frac{P \vee Q \quad \frac{[P]}{P \vee (Q \vee R)} \vee\text{Intro1} \quad \frac{\frac{[Q]}{Q \vee R} \vee\text{Intro1}}{P \vee (Q \vee R)} \vee\text{Intro2}}{P \vee (Q \vee R)} \vee\text{Elim}$$

Our premise $P \vee Q$ splits the proof up into two cases: one where P is true and one where Q is true. Both of these are cases of $P \vee (Q \vee R)$.

Disjunction 3

$$\frac{(P \vee Q) \vee R \quad \frac{\frac{[P \vee Q]^2 \quad \frac{[P]^1}{P \vee (Q \vee R)} \vee\text{Intro1} \quad \frac{[Q]^1}{Q \vee R} \vee\text{Intro1}}{P \vee (Q \vee R)} \vee\text{Intro2} \quad \frac{[R]^2}{Q \vee R} \vee\text{Intro2}}{P \vee (Q \vee R)} \vee\text{Elim}^1}{P \vee (Q \vee R)} \vee\text{Elim}^2}{P \vee (Q \vee R)}$$

This proof works similarly to the previous proofs, but there are two splits: $(P \vee Q) \vee R$ splits the proof into a $P \vee Q$ case and an R case, and then the former case splits into a case of P and a case of Q . In all three cases we apply the $\vee\text{Intro}$ rules to obtain $P \vee (Q \vee R)$.

Disjunction 4

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$$\frac{(P \vee Q) \vee (R \vee P_1) \quad \frac{\frac{[P \vee Q] \quad \frac{[P]}{P \vee P_1} \vee\text{I1}}{(P \vee P_1) \vee (R \vee Q)} \vee\text{I1} \quad \frac{[Q]}{R \vee Q} \vee\text{I2}}{(P \vee P_1) \vee (R \vee Q)} \vee\text{E} \quad \frac{[R \vee P_1] \quad \frac{[R]}{R \vee Q} \vee\text{I1} \quad \frac{[P_1]}{P \vee P_1} \vee\text{I2}}{(P \vee P_1) \vee (R \vee Q)} \vee\text{E}}{(P \vee P_1) \vee (R \vee Q)} \vee\text{E}}{(P \vee P_1) \vee (R \vee Q)}$$

This proof is large, but very systematic: the premise $(P \vee Q) \vee (R \vee P_1)$ splits the proof into two further disjunctions, $(P \vee Q)$ and $(R \vee P_1)$, and then these disjunctions split the cases further into cases of P , Q , R and P_1 . All four of these are cases where the conclusion $(P \vee P_1) \vee (R \vee Q)$ is true.

Disjunction 5

$$\frac{\frac{P \wedge (Q \vee R)}{Q \vee R} \wedge\text{Elim} \quad \frac{\frac{P \wedge (Q \vee R)}{P} \wedge\text{Elim} \quad [Q] \quad \frac{P \wedge (Q \vee R)}{P \wedge Q} \wedge\text{Intro}}{(P \wedge Q) \vee (P \wedge R)} \vee\text{Intro} \quad \frac{\frac{P \wedge (Q \vee R)}{P} \wedge\text{Elim} \quad [R] \quad \frac{P \wedge (Q \vee R)}{P \wedge R} \wedge\text{Intro}}{(P \wedge Q) \vee (P \wedge R)} \vee\text{Intro}}{(P \wedge Q) \vee (P \wedge R)} \vee\text{Elim}$$

Our premise $P \wedge (Q \vee R)$ gives us two sentences to work with: a disjunction $Q \vee R$ and the sentence-letter P alone. $Q \vee R$ splits the proof into a case where Q is true (in the middle) and a case where R is true (on the right). These are, respectively, cases of $P \wedge Q$ and $P \wedge R$, because we get the P for free from our original premise. Therefore both cases are cases of $(P \wedge Q) \vee (P \wedge R)$.

Disjunction 6

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$$\frac{\frac{(P \vee Q) \wedge (P \vee R)}{P \vee Q} \wedge\text{Elim} \quad \frac{[P]^2}{P \vee (Q \wedge R)} \vee\text{Intro} \quad \frac{\frac{(P \vee Q) \wedge (P \vee R)}{P \vee R} \wedge\text{Elim} \quad \frac{[P]^1}{P \vee (Q \wedge R)} \vee\text{Intro} \quad \frac{\frac{[Q]^2 \quad [R]^1}{Q \wedge R} \wedge\text{Intro}}{P \vee (Q \wedge R)} \vee\text{Intro}}{P \vee (Q \wedge R)} \vee\text{Elim}^1}{P \vee (Q \wedge R)} \vee\text{Elim}^2$$

Our premise $(P \vee Q) \wedge (P \vee R)$ is a conjunction, so it gives us two sentences we can use: $P \vee Q$ and $P \vee R$. Working from the bottom up, we can use $P \vee Q$ first to split the proof into a case where P is true and a case where Q is true. In the former case, we can go straight from P to $P \vee (Q \wedge R)$. In the latter case we need to split the proof again: $P \vee R$ splits the proof into a case where P is true (and therefore the conclusion $P \vee (Q \wedge R)$ is true), and a case where R is true. In this final case Q is true as well: we're still part of the case where Q is true, and our assumption of Q gets discharged in the final step of the proof. Because we have both Q and R , we can prove $Q \wedge R$ and therefore have a case where $P \vee (Q \wedge R)$ is true.

Disjunction 7

$$\frac{\frac{(P \wedge Q) \vee (P \wedge R)}{P} \wedge\text{Elim} \quad \frac{\frac{[P \wedge Q]}{Q} \wedge\text{Elim}}{Q \vee R} \vee\text{Intro} \quad \frac{[P \wedge R]}{P} \wedge\text{Elim} \quad \frac{\frac{[P \wedge R]}{R} \wedge\text{Elim}}{Q \vee R} \vee\text{Intro}}{\frac{P \wedge (Q \vee R)}{P \wedge (Q \vee R)} \wedge\text{Intro} \quad \frac{P \wedge (Q \vee R)}{P \wedge (Q \vee R)} \wedge\text{Intro}}{P \wedge (Q \vee R)} \vee\text{Elim}$$

The premise splits this proof into a case where $P \wedge Q$ is true and a case where $P \wedge R$ is true. In the former case, from Q we can derive $Q \vee R$, so it is a case where $P \wedge (Q \vee R)$ is true. Very similarly in the latter case, from R we can derive $Q \vee R$, so $P \wedge (Q \vee R)$ is also true.

Disjunction 8

$$\frac{\frac{P \vee (Q \wedge R)}{P \vee Q} \vee\text{Intro} \quad \frac{[P]}{P \vee R} \vee\text{Intro} \quad \frac{[Q \wedge R]}{Q} \wedge\text{Elim} \quad \frac{[Q \wedge R]}{R} \wedge\text{Elim}}{\frac{P \vee Q}{P \vee Q} \vee\text{Intro} \quad \frac{P \vee R}{P \vee R} \vee\text{Intro} \quad \frac{P \vee Q}{P \vee Q} \vee\text{Intro} \quad \frac{P \vee R}{P \vee R} \vee\text{Intro}}{\frac{(P \vee Q) \wedge (P \vee R)}{(P \vee Q) \wedge (P \vee R)} \wedge\text{Intro} \quad \frac{(P \vee Q) \wedge (P \vee R)}{(P \vee Q) \wedge (P \vee R)} \wedge\text{Intro}}{P \vee (Q \wedge R)} \vee\text{Elim}$$

The premise splits this proof into a case where P is true and a case where $Q \wedge R$ is true. In the first case P leads to both $P \vee Q$ and $P \vee R$, giving the conclusion $(P \vee Q) \wedge (P \vee R)$. In the second case, Q and R lead, respectively, to $P \vee Q$ and $P \vee R$, also giving the conclusion.

Disjunction 9

$$\frac{(P \rightarrow Q) \vee Q \quad [P \rightarrow Q] \quad \frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro}}{P \rightarrow Q} \vee\text{Elim}$$

Our premise $(P \rightarrow Q) \vee Q$ provides one case where $P \rightarrow Q$ is true. This is the conclusion, so we don't need to do anything else. In other case, we can freely assume Q . From this we can apply $\vee\text{Intro}$ to derive $P \rightarrow Q$ in one step, so the conclusion is true in both cases.

Disjunction 10

$$\frac{P \vee Q \quad \frac{[P] \quad [P \rightarrow Q]}{Q} \rightarrow\text{Elim} \quad [Q]}{Q} \vee\text{Elim} \quad \frac{Q}{(P \rightarrow Q) \rightarrow Q} \rightarrow\text{Intro}$$

Our conclusion $(P \rightarrow Q) \rightarrow Q$ is an implication, so we prove it by assuming $P \rightarrow Q$ and providing a proof of Q . Because our premise $P \vee Q$ is a disjunction, we can split the proof into a case where P is true and a case where Q is true. In the case where Q is true, we don't need to do anything. In the case where P is true, we use our assumption of $P \rightarrow Q$ to derive Q by $\rightarrow\text{Elim}$, meaning we have a proof of Q in both cases.

We could also have carried out the proof in a slightly different way and applied $\vee\text{Elim}$ at the very end:

$$\frac{P \vee Q \quad \frac{[P] \quad [P \rightarrow Q]}{Q} \rightarrow\text{Elim} \quad \frac{[Q]}{(P \rightarrow Q) \rightarrow Q} \rightarrow\text{Intro}}{(P \rightarrow Q) \rightarrow Q} \vee\text{Elim}$$

Note that in this proof we apply $\rightarrow\text{Intro}$ in both branches, so the proof is slightly longer. When using $\vee\text{Elim}$ it is often possible to apply it at more than one point of the proof, but applying $\vee\text{Elim}$ as early as possible usually leads to shorter proofs.

Disjunction 11

$$\frac{
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 \frac{
 [Q]
 }{P \rightarrow Q} \rightarrow\text{Intro}
 }{(P \rightarrow Q) \rightarrow (P \rightarrow R)} \rightarrow\text{Elim}
 }{P \rightarrow R}
 }{R} \rightarrow\text{Elim}
 }{[P]} \rightarrow\text{Elim}
 }{[R]} \vee\text{Elim}
 }{[P \vee R]} \vee\text{Elim}
 }{
 \frac{
 \frac{
 R
 }{Q \rightarrow R} \rightarrow\text{Intro}
 }{(P \vee R) \rightarrow (Q \rightarrow R)} \rightarrow\text{Intro}
 }$$

The conclusion $(P \vee R) \rightarrow (Q \rightarrow R)$ is an implication. This means to prove it we need to prove $Q \rightarrow R$, and we can freely assume $P \vee R$. $Q \rightarrow R$ is an implication as well; this means we can freely assume Q and need to prove R . $P \vee R$ splits the proof into two cases: in one case we have P and in the other we have R , and in both cases we need to prove R .

The right case is easy: R is just true. In the left case, proving R from P is more complex. To prove $Q \rightarrow R$, we need to get to R and we can freely assume Q . This is very helpful: Q gives us $P \rightarrow Q$, which then (thanks to our premise $(P \rightarrow Q) \rightarrow (P \rightarrow R)$) gives us $P \rightarrow R$. That, with the P we have, gives us R .

Disjunction 12

$$\frac{
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 \frac{
 [P] \quad [P \rightarrow Q]
 }{Q \vee R} \vee\text{Intro}
 }{(P \rightarrow Q) \vee (P \rightarrow R)} \vee\text{Elim}
 }{
 \frac{
 \frac{
 [P] \quad [P \rightarrow R]
 }{Q \vee R} \vee\text{Intro}
 }{Q \vee R} \vee\text{Elim}
 }{
 \frac{
 Q \vee R
 }{P \rightarrow (Q \vee R)} \rightarrow\text{Intro}
 }$$

Our conclusion is an implication, so we can freely assume P and need to work towards $Q \vee R$. Our premise $(P \rightarrow Q) \vee (P \rightarrow R)$ splits the proof into a case where $P \rightarrow Q$ is true and a case where $P \rightarrow R$ is true. These cases yield, respectively, Q and R (because P is assumed in both cases) and therefore both yield $Q \vee R$.

Disjunction 13

$$\frac{\frac{\frac{[P \vee Q]}{[P]} \quad \frac{\frac{(P \rightarrow Q) \wedge (Q \rightarrow P)}{P \rightarrow Q} \wedge \text{Elim} \quad [P]}{Q} \rightarrow \text{Elim}}{P \wedge Q} \wedge \text{Intro} \quad \frac{\frac{[P \vee Q]}{[Q]} \quad \frac{\frac{(P \rightarrow Q) \wedge (Q \rightarrow P)}{Q \rightarrow P} \wedge \text{Elim} \quad [Q]}{P} \rightarrow \text{Elim}}{P \wedge Q} \wedge \text{Intro}}{P \wedge Q} \vee \text{Elim}}{(P \vee Q) \rightarrow (P \wedge Q)} \rightarrow \text{Intro}$$

Because our conclusion is an implication, we can freely assume $P \vee Q$ and need to prove $P \wedge Q$. $P \vee Q$ splits the proof into a case where P is true and a case where Q is true. Our premise tells us that P implies Q and that Q implies P . This means that in the case where P is true we can derive Q and in the case where Q is true we can derive P . Therefore $P \wedge Q$ is true in both cases.

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Disjunction 14

$$\frac{\frac{\frac{[P]}{P \vee Q} \vee \text{Intro} \quad \frac{(P \vee Q) \rightarrow (P \wedge Q)}{P \wedge Q} \rightarrow \text{Elim}}{\frac{Q}{P \rightarrow Q} \wedge \text{Elim}} \rightarrow \text{Intro} \quad \frac{\frac{[Q]}{P \vee Q} \vee \text{Intro} \quad \frac{(P \vee Q) \rightarrow (P \wedge Q)}{P \wedge Q} \rightarrow \text{Elim}}{\frac{P}{Q \rightarrow P} \wedge \text{Elim}} \rightarrow \text{Intro}}{(P \rightarrow Q) \wedge (Q \rightarrow P)} \wedge \text{Intro}$$

Our conclusion is a conjunction, so there are two proofs we have to make: one proof of $P \rightarrow Q$ and another proof of $Q \rightarrow P$. In the left proof, because we are proving $P \rightarrow Q$, we can start from P and need to work towards Q . P gives us $P \vee Q$, which (thanks to our premise, $(P \vee Q) \rightarrow (P \wedge Q)$) gives us $P \wedge Q$ and therefore Q . On the other side, Q also gives us $P \vee Q$, allowing us to derive P in the same way as in the left proof.

Disjunction 15

$$\frac{\frac{\frac{(Q \rightarrow R) \wedge (Q \vee P)}{Q \vee P} \wedge\text{Elim} \quad \frac{[P] \quad [P \rightarrow Q]}{Q} \vee\text{Elim} \rightarrow\text{Elim}}{Q} \wedge\text{Elim} \quad \frac{(Q \rightarrow R) \wedge (Q \vee P)}{Q \rightarrow R} \wedge\text{Elim} \quad \frac{(Q \rightarrow R) \wedge (Q \vee P)}{Q \vee P} \wedge\text{Elim} \quad \frac{[P] \quad [P \rightarrow Q]}{Q} \vee\text{Elim} \rightarrow\text{Elim}}{R} \wedge\text{Intro} \quad \frac{R \wedge Q}{(P \rightarrow Q) \rightarrow (R \wedge Q)} \rightarrow\text{Intro}}{(P \rightarrow Q) \rightarrow (R \wedge Q)} \rightarrow\text{Intro}$$

Here our conclusion is an implication; we can freely assume $P \rightarrow Q$ and need to derive $R \wedge Q$. To derive $R \wedge Q$ we need to derive R (which we do on the left-hand side) and Q (which we derive on the right-hand side). Our premise is a conjunction, so we can derive two helpful sentences from it: $Q \vee P$ and $Q \rightarrow R$.

On the right-hand side (where we try to derive Q) we make use of $Q \vee P$. We have one case where we can assume Q , and another case where we can assume P . In the latter case we can use our assumption of $P \rightarrow Q$ to derive Q , so Q is true in both cases. On the left-hand side, we prove Q using the same technique we used on the right-hand side, and then use $Q \rightarrow R$ (which we have derived from our premise) to derive R .

5.5 Biconditional

Biconditional 1

$$\frac{\frac{[P] \quad P \leftrightarrow Q}{Q} \leftrightarrow\text{Elim1} \quad \frac{[Q] \quad P \leftrightarrow Q}{P} \leftrightarrow\text{Elim2}}{Q \leftrightarrow P} \leftrightarrow\text{Intro}$$

The conclusion is a biconditional, so there are two proofs we need to give: one from P to Q and one from Q to P . Both of these can be done in one step, because we have $P \leftrightarrow Q$ as a premise. Our initial assumptions of P and Q are discharged in the final step.

Biconditional 2

$$\frac{\frac{[Q] \quad P}{P \leftrightarrow Q} \leftrightarrow\text{Intro} \quad \frac{(P \leftrightarrow Q) \leftrightarrow R}{R} \leftrightarrow\text{Elim1}}{R} \leftrightarrow\text{Elim1} \quad \frac{\frac{[R] \quad (P \leftrightarrow Q) \leftrightarrow R}{P \leftrightarrow Q} \leftrightarrow\text{Elim2} \quad P}{Q} \leftrightarrow\text{Elim1}}{Q \leftrightarrow R} \leftrightarrow\text{Intro}$$

On the left we have a proof from Q to R and on the right we have a proof from R to Q . The right-hand side is fairly straightforward: we can use R with our premise $(P \leftrightarrow Q) \leftrightarrow R$ to obtain $P \leftrightarrow Q$, and we can then use that with our other premise P to obtain Q . The left-hand side has a less intuitive step: we use our assumption of Q (which will be discharged in the final step) and our premise P to go straight to $P \leftrightarrow Q$; when doing this, no assumptions are discharged. $P \leftrightarrow Q$ and our premise $(P \leftrightarrow Q) \leftrightarrow R$ then give us R .

Biconditional 3

$$\frac{\frac{[Q] \quad [P \leftrightarrow Q]}{P} \leftrightarrow\text{E2} \quad \frac{[P] \quad [P \leftrightarrow Q]}{Q} \leftrightarrow\text{E1} \quad \frac{[P] \quad [Q \leftrightarrow P]}{Q} \leftrightarrow\text{E2} \quad \frac{[Q] \quad [Q \leftrightarrow P]}{P} \leftrightarrow\text{E1}}{\frac{Q \leftrightarrow P}{P \leftrightarrow Q} \leftrightarrow\text{I} \quad \frac{P \leftrightarrow Q}{P \leftrightarrow Q} \leftrightarrow\text{I}} \leftrightarrow\text{I}$$

This is a past paper question from 2014. We have to prove a biconditional: on the left we need to go from $P \leftrightarrow Q$ to $Q \leftrightarrow P$, and on the right we need to go in the other direction. Because in both cases we need to prove a biconditional, the proof splits again: we need to prove P from Q and Q from P on the left, and Q from P and P from Q on the right. All four of these can be achieved in one step because of the assumptions we have of $P \leftrightarrow Q$ and $Q \leftrightarrow P$, which are discharged in the final step.

Biconditional 4

$$\frac{\frac{[P]}{P \vee Q} \vee\text{Intro} \quad (P \vee Q) \leftrightarrow Q}{Q} \leftrightarrow\text{Elim} \\ \frac{Q}{P \rightarrow Q} \rightarrow\text{Intro}$$

This proof is a one-way implication; we can freely assume P and need to work towards Q . P gives us $P \vee Q$, and this combined with our premise $(P \vee Q) \leftrightarrow Q$ gives us Q .

Biconditional 5

$$\frac{[P] \quad (P \wedge Q) \leftrightarrow P}{P \wedge Q} \leftrightarrow\text{Elim} \\ \frac{P \wedge Q}{Q} \wedge\text{Elim} \\ \frac{Q}{P \rightarrow Q} \rightarrow\text{Intro}$$

Here we are proving $P \rightarrow Q$, which means we can freely assume P and have to derive Q . P and our premise $(P \wedge Q) \leftrightarrow P$ allow us to derive $P \wedge Q$, which then gives us Q .

Biconditional 6

$$\frac{[P \vee Q] \quad \frac{[P] \quad P \rightarrow Q}{Q} \rightarrow\text{Elim} \quad [Q]}{Q} \vee\text{Elim} \quad \frac{[Q]}{P \vee Q} \vee\text{Intro} \\ \frac{Q}{(P \vee Q) \leftrightarrow Q} \leftrightarrow\text{Intro}$$

Here we need to prove Q from $P \vee Q$ on the left and $P \vee Q$ from Q on the right. The right-hand side is easy: $P \vee Q$ can be derived from Q in a single step. On the left-hand side the proof is more complex. Our assumption of $P \vee Q$ (which is discharged in the final step) splits the proof two cases. In one, Q is true, which is what we need to derive. In the other, P is true; this, together with the premise $P \rightarrow Q$, lets us prove Q .

Biconditional 7

$$\frac{[P \wedge Q]}{P} \wedge\text{Elim} \quad \frac{[P] \quad \frac{[P] \quad P \rightarrow Q}{Q} \rightarrow\text{Elim}}{P \wedge Q} \wedge\text{Intro} \\ \frac{P \quad P \wedge Q}{(P \wedge Q) \leftrightarrow P} \leftrightarrow\text{Intro}$$

The conclusion $(P \wedge Q) \leftrightarrow P$ asks us to provide two proofs: a proof of P from $P \wedge Q$ (this is on the left and is a simple case of $\wedge\text{Elim}$) and a proof of $P \wedge Q$ from P . For this latter proof, we use our assumption of P with the premise $P \rightarrow Q$ to give us Q , and then join that with our assumption of P to obtain $P \wedge Q$.

Biconditional 8

$$\frac{\frac{(P \rightarrow Q) \wedge (Q \rightarrow P)}{P \rightarrow Q} \wedge\text{Elim} \quad [P] \rightarrow\text{Elim} \quad \frac{(P \rightarrow Q) \wedge (Q \rightarrow P)}{Q \rightarrow P} \wedge\text{Elim} \quad [Q] \rightarrow\text{Elim}}{\frac{Q}{P} \leftrightarrow\text{Intro} \quad \frac{P}{Q} \leftrightarrow\text{Intro}} P \leftrightarrow Q$$

This is a past paper question from 2013. Our premise $(P \rightarrow Q) \wedge (Q \rightarrow P)$ gives us two sentences we can use: $P \rightarrow Q$ allows us to prove Q from P on the left-hand side and $Q \rightarrow P$ allows us to prove P from Q on the right-hand side.

Biconditional 9

$$\frac{\frac{[P \wedge Q]}{P} \wedge\text{Elim} \quad \frac{[P \wedge Q]}{Q} \wedge\text{Elim} \quad \frac{P}{P \rightarrow Q} \rightarrow\text{Intro} \quad \frac{Q}{(P \rightarrow Q) \leftrightarrow P} \leftrightarrow\text{Intro}}{\frac{(P \rightarrow Q) \leftrightarrow P}{(P \wedge Q) \rightarrow ((P \rightarrow Q) \leftrightarrow P)} \rightarrow\text{Intro}}$$

The conclusion of this proof, $(P \wedge Q) \rightarrow ((P \rightarrow Q) \leftrightarrow P)$, is a conditional: this means we can assume $P \wedge Q$ and need to obtain $(P \rightarrow Q) \leftrightarrow P$. Obtaining $(P \rightarrow Q) \leftrightarrow P$ turns out to be quite easy. Although we are allowed to assume $P \rightarrow Q$ in our proof of P (on the left) and we are allowed to assume P in our proof of $P \rightarrow Q$ (on the right), we don't need either of these. Both P and $P \rightarrow Q$ follow from $P \wedge Q$ alone.

Biconditional 10

$$\frac{\frac{[P] \quad [(P \rightarrow Q) \leftrightarrow P]}{P \rightarrow Q} \leftrightarrow\text{Elim} \quad [P] \rightarrow\text{Elim} \quad \frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro} \quad \frac{[(P \rightarrow Q) \leftrightarrow P]}{P} \leftrightarrow\text{Elim}}{\frac{Q}{P} \leftrightarrow\text{Intro} \quad \frac{P}{Q} \leftrightarrow\text{Intro}} P \leftrightarrow Q \rightarrow\text{Intro} \quad \frac{((P \rightarrow Q) \leftrightarrow P) \rightarrow (P \leftrightarrow Q)}{((P \rightarrow Q) \leftrightarrow P) \rightarrow (P \leftrightarrow Q)} \rightarrow\text{Intro}$$

The conclusion of this proof is a conditional. This means we can assume $(P \rightarrow Q) \leftrightarrow P$ and need to derive $P \leftrightarrow Q$. To prove $P \leftrightarrow Q$ we need two proofs: a proof of Q from P (on the left) and a proof of P from Q (on the right). On the left, our assumption of P together with our assumption of $(P \rightarrow Q) \leftrightarrow P$ gives us $P \rightarrow Q$, and we can use this with another assumption of P to get Q . On the right, from Q we can prove $P \rightarrow Q$, and this in combination with the assumption of $(P \rightarrow Q) \leftrightarrow P$ allows us to prove P .

Biconditional 11

$$\frac{\frac{\frac{[P]}{P \vee Q} \vee I \quad \frac{[P] \quad ((P \vee Q) \leftrightarrow Q) \leftrightarrow P}{(P \vee Q) \leftrightarrow Q} \leftrightarrow E}{Q} \quad \frac{\frac{[Q] \quad \frac{[Q]}{P \vee Q} \vee I}{(P \vee Q) \leftrightarrow Q} \leftrightarrow I \quad ((P \vee Q) \leftrightarrow Q) \leftrightarrow P}{P} \leftrightarrow I}{P \leftrightarrow Q} \leftrightarrow E$$

For this proof we need to prove Q from assumptions of P on the left-hand side and P from Q on the right-hand side. On the left-hand side, P can be used with the premise $((P \vee Q) \leftrightarrow Q) \leftrightarrow P$ to derive $(P \vee Q) \leftrightarrow Q$. In one step we can prove $P \vee Q$ from P , and that means we can apply \leftrightarrow Elim to derive Q .

On the right-hand side, $(P \vee Q) \leftrightarrow Q$ is fairly easy to derive: we have Q on one side and $P \vee Q$ (which follows from Q in one step) on the other side, which means we can derive $(P \vee Q) \leftrightarrow Q$ without discharging any assumptions. This can be used with the premise $((P \vee Q) \leftrightarrow Q) \leftrightarrow P$ to derive P .

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Biconditional 12

$$\frac{\frac{\frac{[P \wedge Q]}{P} \wedge E \quad \frac{\frac{[P \wedge Q]}{Q} \wedge E \quad \frac{[P \wedge Q]}{P} \wedge E \quad P \rightarrow (Q \leftrightarrow R)}{Q \leftrightarrow R} \rightarrow E}{R} \wedge I}{P \wedge R} \wedge I \quad \frac{\frac{[P \wedge R]}{P} \wedge E \quad \frac{\frac{[P \wedge R]}{R} \wedge E \quad \frac{[P \wedge R]}{P} \wedge E \quad P \rightarrow (Q \leftrightarrow R)}{Q \leftrightarrow R} \rightarrow E}{Q} \wedge I}{P \wedge Q} \wedge I}{(P \wedge Q) \leftrightarrow (P \wedge R)} \leftrightarrow I$$

This is a past paper question from 2012. In order to prove the biconditional $(P \wedge Q) \leftrightarrow (P \wedge R)$, we need to prove $P \wedge R$ from assumptions of $P \wedge Q$ and we need to prove $P \wedge Q$ from assumptions of $P \wedge R$. On the left-hand side, our assumption of $P \wedge Q$ allows us to obtain P . This, combined with our premise $P \rightarrow (Q \leftrightarrow R)$, gives us $Q \leftrightarrow R$. We can use this with Q (which we also obtain from $P \wedge Q$) to obtain R ; this and P give us $P \wedge R$. The right-hand side works the same way, except we work from $Q \leftrightarrow R$ and R to derive Q .

Biconditional 13

$$\frac{\frac{\frac{[P]}{P \vee R} \vee I \quad \frac{[P]}{P \vee R} \vee I}{(P \vee Q) \wedge (P \vee R)} \wedge I \quad \frac{\frac{\frac{[Q \wedge R]}{Q} \wedge E \quad \frac{[Q \wedge R]}{R} \wedge E}{P \vee Q} \vee I \quad \frac{[Q \wedge R]}{P \vee R} \vee I}{(P \vee Q) \wedge (P \vee R)} \wedge I}{(P \vee Q) \wedge (P \vee R)} \vee E \quad \frac{[(P \vee Q) \wedge (P \vee R)]}{P \vee Q} \wedge E \quad \frac{[P]}{P \vee (Q \wedge R)} \vee I \quad \frac{[(P \vee Q) \wedge (P \vee R)]}{P \vee R} \wedge E \quad \frac{[P]}{P \vee (Q \wedge R)} \vee I \quad \frac{\frac{[Q] \quad [R]}{Q \wedge R} \wedge I}{P \vee (Q \wedge R)} \vee I}{P \vee (Q \wedge R)} \vee E}{(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))} \leftrightarrow I$$

This is a past paper question from 2010. It breaks down into two proofs: a proof of $(P \vee Q) \wedge (P \vee R)$ from $P \vee (Q \wedge R)$ and a proof of $P \vee (Q \wedge R)$ from $(P \vee Q) \wedge (P \vee R)$.

On the left-hand side, our assumption of $P \vee (Q \wedge R)$ splits the proof further into a case where P is true and a case where $Q \wedge R$ is true. In the first case P leads to both $P \vee Q$ and $P \vee R$, giving $(P \vee Q) \wedge (P \vee R)$. In the second case, Q and R lead, respectively, to $P \vee Q$ and $P \vee R$, also giving $(P \vee Q) \wedge (P \vee R)$.

On the right-hand side, our assumption of $(P \vee Q) \wedge (P \vee R)$ is a conjunction, giving us $P \vee Q$ and $P \vee R$. $P \vee Q$ splits this part of the proof into a case where P is true and a case where Q is true. In the former case, we can derive $P \vee (Q \wedge R)$ from P . In the latter case we use $P \vee R$ to split the proof again into a case where P is true (and therefore the conclusion $P \vee (Q \wedge R)$ is true), and a case where R is true. In this final case Q is true as well: we're still part of the case where Q is true, and our assumption of Q gets discharged in the final step of the proof. Because we have both Q and R , we can prove $Q \wedge R$ and therefore have a case where $P \vee (Q \wedge R)$ is true.

5.6 Negation

Negation 1

$$\frac{P \quad [\neg P]}{\neg\neg P} \neg\text{Intro}$$

In order to prove $\neg\neg P$ (in other words, to prove that it is not the case that $\neg P$) we start by assuming $\neg P$ and show that it leads to a contradiction. Showing it leads to a contradiction is very easy: $\neg P$ contradicts our premise P . Applying \neg Intro allows us to discharge our assumption of $\neg P$ and prove $\neg\neg P$.

Negation 2

$$\frac{\frac{[P \wedge Q]}{P} \wedge\text{Elim} \quad \neg P}{\neg(P \wedge Q)} \neg\text{Intro}$$

To prove $\neg(P \wedge Q)$, we start by assuming $P \wedge Q$ and try to derive a contradiction from it. $P \wedge Q$ gives us P , which contradicts our premise $\neg P$.

Negation 3

$$\frac{[P] \quad \frac{P \rightarrow \neg P}{\neg P} \rightarrow\text{Elim}}{\neg P} \neg\text{Intro}$$

To prove $\neg P$, we start by assuming P and try to derive a contradiction from it. Our premise $P \rightarrow \neg P$ doesn't contradict P straight away, but because we've assumed P we can apply \rightarrow Elim to obtain $\neg P$; this then contradicts P and allows us to apply \neg Intro.

Negation 4

$$\frac{\frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro} \quad \neg(P \rightarrow Q)}{\neg Q} \neg\text{Intro}$$

To prove $\neg Q$, we start by assuming Q and try to derive a contradiction. We can derive $P \rightarrow Q$ from Q in one step without discharging any assumptions; this then contradicts our premise $\neg(P \rightarrow Q)$.

Negation 5

$$\frac{\frac{[P] \quad [Q]}{P \wedge Q} \wedge\text{Intro} \quad \neg(P \wedge Q)}{\neg Q} \neg\text{Intro}}{P \rightarrow \neg Q} \rightarrow\text{Intro}$$

Our conclusion $P \rightarrow \neg Q$ is a conditional statement, so we assume P and try to derive $\neg Q$. To derive $\neg Q$, we assume Q and try to prove a contradiction. Our premise $\neg(P \wedge Q)$ is a negated statement, so we have a contradiction if we can prove $P \wedge Q$. Because we have assumptions of P and Q , we can indeed derive $P \wedge Q$. This gives us the contradiction we need to apply $\neg\text{Intro}$, derive $\neg Q$ and discharge our assumption of Q . Finally we apply $\rightarrow\text{Intro}$ to discharge our assumption of P and prove our conclusion.

Negation 6

$$\frac{\frac{[P] \quad P \rightarrow Q}{Q} \rightarrow\text{Elim} \quad [\neg Q]}{\neg P} \neg\text{Intro}}{\neg Q \rightarrow \neg P} \rightarrow\text{Intro}$$

Our conclusion $\neg Q \rightarrow \neg P$ is a conditional statement. This means we can assume $\neg Q$ and need to prove $\neg P$. To prove $\neg P$ we can assume P and need to derive a contradiction. Our assumption of P and the premise $P \rightarrow Q$ let us prove Q , which contradicts our assumption of $\neg Q$. P is discharged in the $\neg\text{Intro}$ step, while $\neg Q$ is discharged in the $\rightarrow\text{Intro}$ step.

Negation 7

$$\frac{\frac{[(P \wedge \neg P) \vee (Q \wedge \neg Q)] \quad \frac{\frac{[P \wedge \neg P]}{P} \wedge\text{Elim} \quad \frac{[P \wedge \neg P]}{\neg P} \wedge\text{Elim}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))} \neg\text{Intro} \quad \frac{\frac{[Q \wedge \neg Q]}{Q} \wedge\text{Elim} \quad \frac{[Q \wedge \neg Q]}{\neg Q} \wedge\text{Elim}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))} \neg\text{Intro}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))} \vee\text{Elim}}{\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))} \neg\text{Intro} \quad [(P \wedge \neg P) \vee (Q \wedge \neg Q)] \neg\text{Intro}$$

Our conclusion $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ is a negated statement, so we can prove it by assuming $(P \wedge \neg P) \vee (Q \wedge \neg Q)$ and deriving a contradiction from it. This is a disjunction, so it splits the proof into a case where $P \wedge \neg P$ is true and a case where $Q \wedge \neg Q$ is true.

In both of these cases, it is easy to find a contradiction: in the left-hand case P contradicts $\neg P$ and in the right-hand case Q contradicts $\neg Q$. However, when we apply \neg Intro and derive $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ within each case we can't discharge our assumption of $(P \wedge \neg P) \vee (Q \wedge \neg Q)$, since it appears further down in the proof.

Instead what we have to do after applying \vee Elim (discharging $P \wedge \neg P$ and $Q \wedge \neg Q$) is assume $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ a second time. Then we can apply \neg Intro, derive $\neg((P \wedge \neg P) \vee (Q \wedge \neg Q))$ again and discharge both assumptions of $(P \wedge \neg P) \vee (Q \wedge \neg Q)$.

Negation 8

$$\frac{\frac{\frac{[P]}{P \vee Q} \vee\text{Intro} \quad \neg(P \vee Q)}{\neg P} \neg\text{Intro} \quad \frac{\frac{[Q]}{P \vee Q} \vee\text{Intro} \quad \neg(P \vee Q)}{\neg Q} \neg\text{Intro}}{\neg P \wedge \neg Q} \wedge\text{Intro}}$$

Our conclusion is a conjunction, so we need to provide two proofs: a proof of $\neg P$ and a proof of $\neg Q$. On the left-hand side we assume P and try to derive a contradiction. Our premise $\neg(P \vee Q)$ is a negated statement, so we can get the contradiction we need by deriving $P \vee Q$ from P . Similarly, on the right-hand side we derive $P \vee Q$ from our assumption of Q .

Negation 9

$$\frac{\frac{\frac{[P \wedge Q]}{P} \wedge\text{Elim} \quad [\neg P]}{\neg(P \wedge Q)} \neg\text{Intro} \quad \frac{\frac{[P \wedge Q]}{Q} \wedge\text{Elim} \quad [\neg Q]}{\neg(P \wedge Q)} \neg\text{Intro}}{\neg(P \vee \neg Q)} \vee\text{Elim} \quad \neg(P \wedge Q)}$$

Our premise $\neg P \vee \neg Q$ is a disjunction, which splits the proof into a case where we can assume $\neg P$ and a case where we can assume $\neg Q$. In both cases we need to prove $\neg(P \wedge Q)$, which we do by assuming $P \wedge Q$ and showing it leads to a contradiction. From $P \wedge Q$ we can prove P , which contradicts $\neg P$ on the left-hand side, and Q , which contradicts $\neg Q$ on the right-hand side.

Negation 10

$$\frac{\frac{[P] \quad \neg P}{Q} \neg\text{Elim}}{P \rightarrow Q} \rightarrow\text{Intro}$$

To prove $P \rightarrow Q$ we start from an assumption of P and need to derive Q . P contradicts our premise $\neg P$, and this allows us to apply $\neg\text{Elim}$ and prove Q . No assumptions are discharged until the next step, where $\rightarrow\text{Intro}$ discharges P .

Negation 11

$$\frac{\frac{P \wedge \neg P}{P} \wedge\text{Elim} \quad \frac{P \wedge \neg P}{\neg P} \wedge\text{Elim}}{Q} \neg\text{Elim}$$

Here, as in the previous problem, we prove Q from a contradiction. This is provided from our premise alone: from $P \wedge \neg P$ we can prove both P and $\neg P$, which contradict each other.

Negation 12

$$\frac{\frac{P \vee Q \quad \frac{[P] \quad [\neg P]}{Q} \neg\text{Elim}}{[Q]} \vee\text{Elim}}{\frac{Q}{\neg P \rightarrow Q} \rightarrow\text{Intro}}$$

This is similar to a past paper problem from 2013. In order to prove $\neg P \rightarrow Q$, we can freely assume $\neg P$ and need to prove Q . Our premise $P \vee Q$ gives us two cases to consider: one we assume Q (which is great, because that's what we need to prove) and one where we assume P . P contradicts our assumption of $\neg P$, allowing us to derive Q .

Negation 13

$$\frac{\frac{\frac{P \wedge \neg Q}{P} \wedge\text{Elim} \quad P \rightarrow Q}{Q} \rightarrow\text{Elim} \quad \frac{P \wedge \neg Q}{\neg Q} \wedge\text{Elim}}{R} \neg\text{Elim}$$

In order to prove R (which doesn't appear in any of our premises) we need to find a contradiction. $P \wedge \neg Q$ and $P \rightarrow Q$ together provide one: $P \wedge \neg Q$ lets us prove P , and with this and $P \rightarrow Q$ we can prove Q . This contradicts the $\neg Q$ which also follows from $P \wedge \neg Q$.

Negation 14

$$\frac{\frac{P \vee Q \quad \frac{[P] \quad \frac{P \leftrightarrow Q}{Q} \leftrightarrow\text{E}}{P \wedge Q} \wedge\text{I}}{P \wedge Q} \vee\text{E} \quad \frac{[Q] \quad \frac{P \leftrightarrow Q}{P} \leftrightarrow\text{E}}{P \wedge Q} \wedge\text{I}}{R} \neg\text{E} \quad \neg(P \wedge Q)$$

R doesn't appear in any of our premises, so we prove it by finding a contradiction. Of our three premises, $\neg(P \wedge Q)$ is a negated statement, so it makes sense to try to prove $P \wedge Q$ in order to find the contradiction we need. $P \vee Q$ splits the proof into two cases: one where P is true and one where Q is true. In both of these cases we apply the other premise $P \leftrightarrow Q$ to obtain the other sentence-letter and hence derive $P \wedge Q$. This contradicts $\neg(P \wedge Q)$, allowing us to derive R .

Negation 15

$$\frac{[\neg P] \quad \neg\neg P}{P} \neg\text{Elim}$$

Here we cannot provide a proof of P directly. Instead, we prove P by assuming $\neg P$ and showing it leads to a contradiction. $\neg P$ contradicts our premise $\neg\neg P$, meaning we can apply $\neg\text{Elim}$; this discharges our assumption of $\neg P$ and provides a proof of P .

Negation 16

$$\frac{\frac{[P]^1}{P \vee \neg P} \vee\text{Intro} \quad \frac{[\neg(P \vee \neg P)]^2}{\neg P} \neg\text{Intro}^1}{\frac{[\neg(P \vee \neg P)]^2}{P \vee \neg P} \vee\text{Intro} \quad \frac{[\neg(P \vee \neg P)]^2}{P \vee \neg P} \neg\text{Elim}^2}{} \neg\text{Intro}^1$$

This proof is counter-intuitive, relying on a special technique. Our conclusion $P \vee \neg P$ is a disjunction, but we cannot show it is true by proving either disjunct: we have no proof of P and no proof of $\neg P$. Instead, we must assume that $P \vee \neg P$ is false (in other words, we assume $\neg(P \vee \neg P)$) and derive a contradiction from that.

But we still have a problem: even if we assume $\neg(P \vee \neg P)$, what do we have which it contradicts? This proof has no premises. We have to rely on a slightly counter-intuitive trick. We assume $\neg(P \vee \neg P)$ and try to derive a contradiction from it, but initially we also assume P . From P we can prove $P \vee \neg P$, which contradicts $\neg(P \vee \neg P)$.

But we can't go right ahead and apply $\neg\text{Elim}$ to prove $P \vee \neg P$, because that would leave our assumption of P undischarged. So instead what we do is use the contradiction to apply $\neg\text{Intro}$, discharging P and proving $\neg P$.

From $\neg P$, we can apply $\vee\text{Intro}$ to prove $P \vee \neg P$ again. Assuming $\neg(P \vee \neg P)$ leads to a contradiction again, but this time our only undischarged assumption is $\neg(P \vee \neg P)$ itself. We're free to apply $\neg\text{Elim}$, discharge our two assumptions of $\neg(P \vee \neg P)$ and prove $P \vee \neg P$.

The following is a possible alternative proof:

$$\frac{\frac{[P]^1}{P \vee \neg P} \vee\text{Intro} \quad \frac{[\neg(P \vee \neg P)]^3}{\neg P} \neg\text{Intro}^1}{\frac{[\neg(P \vee \neg P)]^3}{P \vee \neg P} \vee\text{Intro} \quad \frac{[\neg(P \vee \neg P)]^3}{P} \neg\text{Elim}^2}{} \neg\text{Elim}^3$$

This kind of structure is possible for many indirect proofs, but it tends to produce longer proofs than the technique above.

Negation 17

$$\frac{\frac{\frac{[\neg P]}{\neg P \vee \neg Q} \vee\text{Intro} \quad \neg(\neg P \vee \neg Q)}{P} \neg\text{Elim} \quad \frac{\frac{[\neg Q]}{\neg P \vee \neg Q} \vee\text{Intro} \quad \neg(\neg P \vee \neg Q)}{Q} \neg\text{Elim}}{P \wedge Q} \wedge\text{Intro}}$$

In order to prove $P \wedge Q$ we need to provide a proof of P and a proof of Q . On the left-hand side we can't prove P directly, so instead we assume $\neg P$ and show that this leads to a contradiction. From $\neg P$ we can derive $\neg P \vee \neg Q$, which contradicts our premise $\neg(\neg P \vee \neg Q)$. The contradiction allows us to apply $\neg\text{Elim}$, discharge our assumption of $\neg P$ and derive P . Similarly on the right-hand side we assume $\neg Q$ and show it leads to a contradiction in order to derive Q .

We could also prove $P \wedge Q$ by assuming $\neg(P \wedge Q)$ and deriving a contradiction from it. This proof is slightly longer: we have to assume P and Q as well as $\neg(P \wedge Q)$, and use three contradictions to discharge these three assumptions. This alternate proof is shown below:

$$\frac{\frac{\frac{[P] \quad [Q]}{P \wedge Q} \wedge\text{Intro} \quad [\neg(P \wedge Q)]}{\neg P} \neg\text{Intro} \quad \frac{\frac{\frac{\neg P}{\neg P \vee \neg Q} \vee\text{Intro} \quad \neg(\neg P \vee \neg Q)}{\neg Q} \neg\text{Intro}}{\neg P \vee \neg Q} \vee\text{Intro} \quad \neg(\neg P \vee \neg Q)}{P \wedge Q} \neg\text{Elim}}$$

Negation 18

$$\frac{\frac{\frac{[\neg P]}{\neg P \vee \neg Q} \vee\text{I} \quad [\neg(\neg P \vee \neg Q)]}{P} \neg\text{E} \quad \frac{[Q]}{P \wedge Q} \wedge\text{I} \quad \neg(P \wedge Q)}{\neg Q} \neg\text{I} \quad \frac{\frac{\frac{\neg Q}{\neg P \vee \neg Q} \vee\text{I} \quad [\neg(\neg P \vee \neg Q)]}{\neg P \vee \neg Q} \neg\text{E}}$$

This is a past paper question from 2015. We want to try to prove $\neg P \vee \neg Q$, but our premise $\neg(P \wedge Q)$ doesn't give a direct proof of either $\neg P$ or $\neg Q$. We will need to assume $\neg(\neg P \vee \neg Q)$ and show it leads to a contradiction.

First we can assume $\neg P$. This lets us derive $\neg P \vee \neg Q$, which contradicts our assumption of $\neg(\neg P \vee \neg Q)$. Applying $\neg\text{Elim}$, we discharge our assumption of $\neg P$ and prove P .

P and $\neg(P \wedge Q)$ together imply $\neg Q$. Assuming Q gives us $P \wedge Q$, which contradicts our premise $\neg(P \wedge Q)$ and allows us to apply $\neg\text{Intro}$, discharge Q

and prove $\neg Q$. From this we can prove $\neg P \vee \neg Q$ again, which provides another contradiction with our assumption of $\neg(\neg P \vee \neg Q)$. Finally we discharge all of our assumptions of $\neg(\neg P \vee \neg Q)$ and derive $\neg P \vee \neg Q$.

Negation 19

$$\frac{\frac{\frac{[P] \quad [\neg P]}{Q} \neg\text{Elim}}{P \rightarrow Q} \rightarrow\text{Intro}}{P} \neg(P \rightarrow Q) \neg\text{Elim}}{P} \neg\text{Elim}$$

The premise $\neg(P \rightarrow Q)$ doesn't allow us to prove P directly. Instead, we assume $\neg P$ and try to derive a contradiction from it. What kind of contradiction should we be looking for? Our premise, $\neg(P \rightarrow Q)$, is a negated sentence, so we will have a contradiction if we manage to prove $P \rightarrow Q$.

Proving $P \rightarrow Q$ from an assumption of $\neg P$ is something we've already done in problem 9. To prove $P \rightarrow Q$, we can assume P and need to derive Q . P and $\neg P$ together is a contradiction, allowing us to apply $\neg\text{Elim}$ and derive Q . Applying $\rightarrow\text{Intro}$ then gives us $P \rightarrow Q$, discharging P and providing a sentence which contradicts $\neg(P \rightarrow Q)$. We can then apply $\neg\text{Elim}$, discharge $\neg P$ and prove P .

Negation 20

$$\frac{\frac{\frac{[P] \quad [\neg P]}{Q} \neg\text{Elim}}{P \rightarrow Q} \rightarrow\text{Intro}}{P} (P \rightarrow Q) \rightarrow P \rightarrow\text{Elim}}{P} [\neg P] \neg\text{Elim} \neg\text{Elim}$$

It might be surprising that we need to use a negation rule in this proof: neither the premise $(P \rightarrow Q) \rightarrow P$ nor the conclusion P have any negation symbols in them. But we cannot prove P directly from $(P \rightarrow Q) \rightarrow P$; again, we need to assume $\neg P$ and show that it leads to a contradiction.

Assuming $\neg P$ allows us to derive $P \rightarrow Q$: this is because assuming P (which gets discharged by $\rightarrow\text{Intro}$) allows us to apply $\neg\text{Elim}$ and derive Q . Having proved $P \rightarrow Q$, our premise $(P \rightarrow Q) \rightarrow P$ allows us to derive P . With P and our assumption of $\neg P$, we have a contradiction. This means we can discharge our assumptions of $\neg P$ and derive P .

Negation 21

$$\frac{\frac{[P] \quad P \leftrightarrow \neg\neg Q}{\neg\neg Q} \leftrightarrow\text{Elim} \quad \frac{[Q] \quad [\neg Q]}{\neg\neg Q} \neg\text{Intro} \quad \frac{P \leftrightarrow \neg\neg Q}{P} \leftrightarrow\text{Elim}}{\frac{Q}{P} \neg\text{Elim} \quad \frac{P}{P} \leftrightarrow\text{Intro}} \leftrightarrow\text{Intro} \quad P \leftrightarrow Q$$

This is a past paper question from 2011. In order to prove $P \leftrightarrow Q$, we need to provide a proof of Q from assumptions of P and a proof of P from assumptions of Q .

On the left-hand side, we can use our assumption of P with the premise $P \leftrightarrow \neg\neg Q$ to derive $\neg\neg Q$. From $\neg\neg Q$ we can prove $\neg Q$ indirectly: an assumption of $\neg Q$ contradicts $\neg\neg Q$, so we can discharge $\neg Q$ and derive Q .

On the right-hand side, the premise $P \leftrightarrow \neg\neg Q$ only helps us if we have a proof of $\neg\neg Q$. Fortunately we can derive this from Q : assuming $\neg Q$ leads to a contradiction, so we can apply $\neg\text{Intro}$, discharge the $\neg Q$ and put an extra negation symbol on the front. Having $\neg\neg Q$ then allows us to prove P .

Negation 22

$$\frac{\frac{[P] \quad [\neg P]}{Q} \neg\text{Elim} \quad \frac{P \rightarrow Q}{P \rightarrow Q} \rightarrow\text{Intro} \quad (P \rightarrow Q) \rightarrow Q}{Q} \rightarrow\text{Elim} \quad \frac{P}{\neg Q \rightarrow P} \rightarrow\text{Intro} \quad [\neg Q] \neg\text{Elim}}{\neg Q \rightarrow P} \rightarrow\text{Intro}$$

Our conclusion is a conditional statement, so we assume $\neg Q$ and try to prove P . Our premise $(P \rightarrow Q) \rightarrow Q$ gives us no way of proving P directly, so we have to assume $\neg P$ and try to derive a contradiction.

Because our assumption $\neg Q$ is a negated statement, it makes sense to try to prove Q in order to contradict it. Our premise is $(P \rightarrow Q) \rightarrow Q$, so we know we can prove Q if we can provide a proof of $P \rightarrow Q$.

We do this by assuming P ; from this and our other assumption of $\neg P$ we can derive Q by $\neg\text{Elim}$. Applying $\rightarrow\text{Intro}$ for the first time discharges our assumption of P and lets us prove $P \rightarrow Q$; we then make use of our premise to derive Q , which contradicts our assumption of $\neg Q$. Applying $\neg\text{Elim}$ a second time discharges our assumption of $\neg P$ and lets us prove P . Finally, we apply $\rightarrow\text{Intro}$ a second time to discharge our assumption of $\neg Q$ and prove the conclusion $\neg Q \rightarrow P$.

Negation 23

$$\frac{[P \vee Q] \quad \frac{[P \vee Q] \quad \frac{[P] \quad \frac{\neg P \wedge \neg Q}{\neg P} \wedge\text{Elim}}{\neg(P \vee Q)} \neg\text{Elim}}{\neg(P \vee Q)} \vee\text{Elim}}{\neg(P \vee Q)} \neg\text{Intro}}{\neg(P \vee Q)} \neg\text{Intro}$$

This is a past paper question from 2014. To show $\neg(P \vee Q)$ we start by assuming $P \vee Q$ and try to derive a contradiction; this assumption of $P \vee Q$ splits the proof into a case where we can assume P and a case where we can assume Q . In both of these cases, contradictions are easy to get: our premise $\neg P \wedge \neg Q$ gives us $\neg P$ (which contradicts P on the left-hand side) and also gives us $\neg Q$ (which contradicts Q on the right-hand side).

These contradictions allow us to apply $\neg\text{Elim}$ and prove $\neg(P \vee Q)$, but they don't allow us to discharge our assumption of $P \vee Q$, because it appears in a different branch of the proof. What we need to do is assume $P \vee Q$ again after applying $\vee\text{Elim}$. Then we have another contradiction which lets us discharge both assumptions of $P \vee Q$ and prove $\neg(P \vee Q)$.

Negation 24

$$\frac{\frac{[P] \quad \frac{P \vee (P \rightarrow Q)}{P \vee (P \rightarrow Q)} \vee\text{I}}{\neg P} \neg\text{I}}{[P]} \neg\text{E}}{\frac{\frac{Q}{P \rightarrow Q} \rightarrow\text{I}}{P \vee (P \rightarrow Q)} \vee\text{I}}{P \vee (P \rightarrow Q)} \neg\text{E}} \neg\text{E}$$

This is another indirect proof of a disjunction: by assuming $\neg(P \vee (P \rightarrow Q))$, the negation of what we want to prove, we try to derive a contradiction. We start by assuming P , which lets us prove $P \vee (P \rightarrow Q)$, and therefore gives rise to a contradiction. This contradiction allows us to apply $\neg\text{Intro}$, discharge our assumption of P and derive $\neg P$.

From $\neg P$, it is possible to derive $P \rightarrow Q$: assuming P for a second time gives us a contradiction and allows us to derive Q , and we can then apply $\rightarrow\text{Intro}$ to derive $P \rightarrow Q$ and discharge this second assumption of P . $P \rightarrow Q$ lets us prove $P \vee (P \rightarrow Q)$, so we have a contradiction again. This second contradiction allows us to discharge our assumptions of $\neg(P \vee (P \rightarrow Q))$ and prove $P \vee (P \rightarrow Q)$.

Negation 25

$$\frac{\frac{\frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro}}{(P \rightarrow Q) \vee (Q \rightarrow R)} \vee\text{Intro} \quad [\neg((P \rightarrow Q) \vee (Q \rightarrow R))] \neg\text{Intro} \quad [Q] \neg\text{Elim}}{\neg Q} \neg\text{Intro} \quad [Q] \neg\text{Elim}$$

$$\frac{\frac{\frac{R}{Q \rightarrow R} \rightarrow\text{Intro}}{(P \rightarrow Q) \vee (Q \rightarrow R)} \vee\text{Intro} \quad [\neg((P \rightarrow Q) \vee (Q \rightarrow R))] \neg\text{Elim}}{(P \rightarrow Q) \vee (Q \rightarrow R)} \neg\text{Elim}$$

We prove our conclusion $(P \rightarrow Q) \vee (P \rightarrow R)$ by assuming its negation and trying to derive a contradiction, but this time the very first thing we do is assume Q .

If Q is true, which we assume initially, we can derive $P \rightarrow Q$ and then we can derive $(P \rightarrow Q) \vee (P \rightarrow R)$. We then get a contradiction, so we discharge our assumption of Q and prove $\neg Q$. From $\neg Q$ we can derive $Q \rightarrow R$ and then $(P \rightarrow Q) \vee (P \rightarrow R)$ again, so our assumption of $\neg((P \rightarrow Q) \vee (P \rightarrow R))$ still leads to a contradiction. This means we can discharge our assumptions of $\neg((P \rightarrow Q) \vee (P \rightarrow R))$ and prove $(P \rightarrow Q) \vee (P \rightarrow R)$.

The proof would still have worked if we had initially assumed $P \rightarrow Q$: this is because from $\neg(P \rightarrow Q)$ we can derive $\neg Q$, from which we can then prove $Q \rightarrow R$. Carrying out the proof this way is a tiny bit longer (requiring eight steps instead of seven), but it is reliable: to prove $\phi \vee \psi$ in general by indirect proof, it will always work to start by assuming ϕ . The alternate proof is shown below:

$$\begin{array}{c}
\frac{[P \rightarrow Q]}{(P \rightarrow Q) \vee (Q \rightarrow R)} \vee\text{Intro} \quad \frac{[\neg((P \rightarrow Q) \vee (Q \rightarrow R))]}{\neg(P \rightarrow Q)} \neg\text{Intro} \quad \frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro}}{\frac{\neg(P \rightarrow Q)}{\neg Q} \neg\text{Intro} \quad \frac{[Q]}{P \rightarrow Q} \rightarrow\text{Intro}}{\frac{R}{Q \rightarrow R} \rightarrow\text{Intro}} \neg\text{Intro} \quad [Q] \neg\text{Elim}} \\
\frac{\frac{R}{Q \rightarrow R} \rightarrow\text{Intro}}{(P \rightarrow Q) \vee (Q \rightarrow R)} \vee\text{Intro} \quad \frac{[\neg((P \rightarrow Q) \vee (Q \rightarrow R))]}{(P \rightarrow Q) \vee (Q \rightarrow R)} \neg\text{Elim}}
\end{array}$$

Negation 26

66

$$\begin{array}{c}
\frac{R \vee \neg Q}{\frac{[R]^1 \quad \frac{\neg R \wedge \neg Q_2}{\neg R} \wedge\text{E}}{\neg Q} \neg\text{E}} \neg\text{E} \quad \frac{[\neg Q]^1}{P} \vee\text{E}^1 \quad \frac{[\neg P]^2 \quad \neg P \rightarrow Q}{Q} \rightarrow\text{E}}{\frac{P}{Q_1 \vee Q_2} \rightarrow\text{E}^2 \quad \frac{P \rightarrow (Q_1 \vee Q_2)}{Q_1} \rightarrow\text{E}} \rightarrow\text{E} \\
\frac{[Q_2]^3 \quad \frac{\neg R \wedge \neg Q_2}{\neg Q_2} \wedge\text{E}}{Q_1} \neg\text{E} \quad \frac{[Q_1]^3 \quad Q_1}{Q_1} \vee\text{E}^3
\end{array}$$

This is a past paper question from 2010. Although it contains many different sentence letters, it is possible to work through it methodically.

First of all, we can use the premise $\neg R \wedge \neg Q_2$ to derive $\neg R$ and $\neg Q_2$. $\neg R$ is useful because one of our other premises is $R \vee \neg Q$; we can use this to derive $\neg Q$ by \neg -Elim and \vee -Elim.

This allows us to establish P by indirect proof. If we assume $\neg P$, the premise $\neg P \rightarrow Q$ allows us to derive Q , but because we've already proved $\neg Q$ we have a contradiction. Therefore we can discharge $\neg P$ and derive P . With P , we can use our last premise $P \rightarrow (Q_1 \vee Q_2)$ to derive $Q_1 \vee Q_2$. Finally we use our proof of $\neg Q_2$ to derive Q_1 using \neg -Elim and \vee -Elim.

Negation 27

$$\begin{array}{c}
 \frac{[P] \quad \frac{P \rightarrow (Q \vee R)}{Q \vee R} \rightarrow E}{\frac{\frac{\frac{[Q] \quad P \rightarrow Q \rightarrow I}{(P \rightarrow Q) \vee (P \rightarrow R)} \vee I \quad \frac{\frac{[R] \quad P \rightarrow R \rightarrow I}{(P \rightarrow Q) \vee (P \rightarrow R)} \vee I}{(P \rightarrow Q) \vee (P \rightarrow R)} \vee E}{\neg((P \rightarrow Q) \vee (P \rightarrow R))} \neg I \quad [P] \quad \neg E}{\neg P} \neg E} \\
 \frac{\frac{Q}{P \rightarrow Q} \rightarrow I}{(P \rightarrow Q) \vee (P \rightarrow R)} \vee I \quad \frac{[\neg((P \rightarrow Q) \vee (P \rightarrow R))]}{(P \rightarrow Q) \vee (P \rightarrow R)} \neg E}{(P \rightarrow Q) \vee (P \rightarrow R)} \vee E
 \end{array}$$

This proof turns out to be a lot nastier than it might look at first. The premise $P \rightarrow (Q \vee R)$ and an assumption of P give us $Q \vee R$, meaning we work with one case where we can assume Q and another case where we can assume R . Both of these assumptions bring us easily to $(P \rightarrow Q) \vee (P \rightarrow R)$, but even after applying \vee Elim we haven't discharged our assumption of P .

This turns out to be another proof where we prove a disjunction indirectly. If we assume $\neg((P \rightarrow Q) \vee (P \rightarrow R))$, we get a contradiction and we can derive $\neg P$ (discharging our assumption of P). From $\neg P$ we can prove $P \rightarrow Q$, which gets us back to our conclusion $(P \rightarrow Q) \vee (P \rightarrow R)$ and contradicts our assumption of $\neg((P \rightarrow Q) \vee (P \rightarrow R))$. Applying \neg Elim a final time discharges our assumptions of $\neg((P \rightarrow Q) \vee (P \rightarrow R))$ and lets us prove $(P \rightarrow Q) \vee (P \rightarrow R)$.

Negation 28

$$\frac{\frac{\frac{[\neg P]^1}{\neg P \vee \neg Q} \vee I \quad [\neg(\neg P \vee \neg Q)]^3}{P} \neg E^1 \quad [Q]^2}{\frac{P \wedge Q}{\neg P \vee \neg Q} \wedge I \quad [\neg(P \wedge Q)]^7}{\neg I^2} \neg I^2}{\frac{\frac{\neg Q}{\neg P \vee \neg Q} \vee I \quad [\neg(\neg P \vee \neg Q)]^3}{\neg P \vee \neg Q} \neg E^3 \quad [\neg P \vee \neg Q]^7}{\frac{\frac{[P \wedge Q]^4}{P} \wedge E \quad [\neg P]^6}{\neg(P \wedge Q)} \neg I^4 \quad \frac{[P \wedge Q]^5}{Q} \wedge E \quad [\neg Q]^6}{\neg(P \wedge Q)} \neg I^5}{\neg(P \wedge Q)} \vee E^6} \leftrightarrow I^7}{\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)} \leftrightarrow I^7$$

This is a past paper question from 2010. It breaks down into two proofs: a proof of $\neg P \vee \neg Q$ from $\neg(P \wedge Q)$ and a proof of $\neg(P \wedge Q)$ from $\neg P \vee \neg Q$.

On the left-hand side we want to try to prove $\neg P \vee \neg Q$, but our assumption $\neg(P \wedge Q)$ doesn't give a direct proof of either $\neg P$ or $\neg Q$. We need to assume $\neg(\neg P \vee \neg Q)$ and show it leads to a contradiction. First we can assume $\neg P$. This lets us derive $\neg P \vee \neg Q$, which contradicts our assumption of $\neg(\neg P \vee \neg Q)$.

Applying \neg -Elim, we discharge our assumption of $\neg P$ and prove P . P and $\neg(P \wedge Q)$ together imply $\neg Q$. Assuming Q gives us $P \wedge Q$, which contradicts our assumption $\neg(P \wedge Q)$ and allows us to apply \neg -Intro, discharge Q and prove $\neg Q$. From this we can prove $\neg P \vee \neg Q$ again, which provides another contradiction with our assumption of $\neg(\neg P \vee \neg Q)$. Finally we discharge all of our assumptions of $\neg(\neg P \vee \neg Q)$ and derive $\neg P \vee \neg Q$.

On the right-hand side, the proof is slightly easier. Our assumption $\neg P \vee \neg Q$ is a disjunction, splitting the proof into a case where we can assume $\neg P$ and a case where we can assume $\neg Q$. In both cases we need to prove $\neg(P \wedge Q)$, which we do by assuming $P \wedge Q$ and showing it leads to a contradiction. From $P \wedge Q$ we can prove P , which contradicts $\neg P$ in one case, and Q , which contradicts $\neg Q$ in the other cases. In both cases we discharge $P \wedge Q$ and derive $\neg(P \wedge Q)$.

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Bonus challenge 1

It is possible to derive $\neg P$ from $\neg\neg\neg P$ using only one application of \neg Elim, similar to the proof above of P from $\neg\neg P$:

$$\frac{[\neg\neg P] \quad \neg\neg\neg P}{\neg P} \neg\text{Elim}$$

Without \neg Elim, a proof is still possible. We start by assuming P and $\neg P$. Together they give us a contradiction which lets us discharge $\neg P$ and prove $\neg\neg P$. This in turn contradicts our premise $\neg\neg\neg P$, letting us discharge P and prove $\neg P$.

$$\frac{\frac{[P]^2 \quad [\neg P]^1}{\neg\neg P} \neg\text{Intro}^1 \quad \neg\neg\neg P}{\neg P} \neg\text{Intro}^2$$

Similarly, with \neg Intro we can derive $\neg\neg P$ from P in only a single step, but a longer proof is still possible using only \neg Elim. We start by assuming $\neg\neg P$ and $\neg\neg\neg P$, which together give us a contradiction. This lets us discharge $\neg\neg P$ and derive $\neg P$, which contradicts our premise P . This contradiction lets us discharge $\neg\neg\neg P$ and prove $\neg\neg P$.

$$\frac{P \quad \frac{[\neg\neg P]^1 \quad [\neg\neg\neg P]^2}{\neg P} \neg\text{Elim}^2}{\neg\neg P} \neg\text{Elim}^1$$

In fact, any proof involving \neg Intro can be replaced with a larger proof involving \neg Elim. Removing \neg Intro would not make the system of Natural Deduction any weaker. This concept is explored further in one of the additional challenges at the end of this pack. However, it is not always possible to do the reverse and replace \neg Elim by \neg Intro. Without \neg Elim we cannot (for example) prove P from $\neg\neg P$.

Bonus challenge 2

One strategy for deriving $P \wedge Q$ from $\neg\neg P \wedge \neg\neg Q$ is to provide two separate proofs of P and Q . Both of these are indirect proofs where we assume $\neg P$ or $\neg Q$ and derive a contradiction. This strategy provides a proof of only five steps, satisfying the first constraint:

$$\frac{\frac{[\neg P]^1 \quad \frac{\neg\neg P \wedge \neg\neg Q}{\neg\neg P} \wedge\text{Elim}}{P} \neg\text{Elim}^1 \quad \frac{[\neg Q]^2 \quad \frac{\neg\neg P \wedge \neg\neg Q}{\neg\neg Q} \wedge\text{Elim}}{Q} \neg\text{Elim}^2}{P \wedge Q} \wedge\text{Intro}}$$

To satisfy the second constraint, we must provide a proof where $\neg\text{Elim}$ only discharges assumptions in the final step of the proof. Since we probably will want to discharge assumptions using $\neg\text{Elim}$, we will have to make $\neg\text{Elim}$ our last step of the proof.

Since the very last line of the proof will be $P \wedge Q$, our conclusion, we know this final $\neg\text{Elim}$ step can discharge assumptions of $\neg(P \wedge Q)$. Since we can't use $\neg\text{Elim}$ to discharge anything else, we can't assume $\neg P$ or $\neg Q$ and discharge them later. We must assume P and Q at the start of the proof and derive $P \wedge Q$.

We can then produce three contradictions in a row. The first contradiction with $\neg(P \wedge Q)$ lets us derive $\neg P$, discharging P . This contradicts $\neg\neg P$ and lets us derive $\neg Q$, discharging Q . This contradicts $\neg\neg Q$, which lets us discharge $\neg(P \wedge Q)$ and derive our conclusion.

This means we have at the following proof, satisfying the second constraint:

$$\frac{\frac{\frac{[P]^1 \quad [Q]^2}{P \wedge Q} \wedge\text{Intro} \quad [\neg(P \wedge Q)]^3}{\neg P} \neg\text{Intro}^1 \quad \frac{\frac{\neg\neg P \wedge \neg\neg Q}{\neg\neg P} \wedge\text{Elim}}{\neg P} \neg\text{Intro}^2 \quad \frac{\neg\neg P \wedge \neg\neg Q}{\neg\neg Q} \wedge\text{Elim}}{\neg Q} \neg\text{Elim}^3}{P \wedge Q} \neg\text{Elim}$$

This is a specific instance of a general result in propositional logic. Whenever we have to provide a proof of ϕ , we can always do this by assuming $\neg\phi$, showing that it leads to a contradiction and applying $\neg\text{Elim}$ at the end. If we do this, we will never need to discharge any other assumptions using $\neg\text{Elim}$ anywhere else in the proof. However, this proof might not be the shortest possible proof of ϕ .

5.7 Universal quantifier

Universal 1

$$\frac{\frac{\forall x Px}{Pa} \forall\text{Elim}}{\forall y Py} \forall\text{Intro}$$

This proof involves one simple application of each of the two universal quantifier rules. The first step takes us from a universal statement (our premise $\forall x Px$) to a specific statement (Pa). In order to use the universal introduction rule, and go from a specific statement about one constant to a general statement, that constant needs to be arbitrary: it needs to be possible for the proof still to work if we replaced the constant with any other constant. This is the case here: our constant a is arbitrary. It appears in no undischarged assumptions in our proof (in fact, no assumptions at all). This means we are free to apply the $\forall\text{Intro}$ rule and prove $\forall y Py$.

Universal 2

$$\frac{\frac{[Pa]}{Pa \rightarrow Pa} \rightarrow\text{Intro}}{\forall x (Px \rightarrow Px)} \forall\text{Intro}$$

Working from the bottom upwards, we know we need to prove a universal claim $\forall x (Px \rightarrow Px)$. We can derive that universal claim from a specific (but arbitrary) case $Pa \rightarrow Pa$. This is easy to prove: we can assume Pa , and then move to $Pa \rightarrow Pa$ (also discharging Pa) by applying $\rightarrow\text{Intro}$. Our move from $Pa \rightarrow Pa$ to $\forall x (Px \rightarrow Px)$ is justified because, at the time we apply the $\forall\text{Intro}$ rule, a does not appear in any undischarged assumptions in the proof of $Pa \rightarrow Pa$.

Universal 3

$$\frac{\frac{[Pa] \frac{\forall x (Pa \rightarrow Qx)}{Pa \rightarrow Qb} \forall\text{Elim}}{Qb} \rightarrow\text{Elim}}{\frac{Qb}{\forall z Qz} \forall\text{Intro}} \rightarrow\text{Intro}$$

Our conclusion $Pa \rightarrow \forall z Qz$ is an implication. This means we can freely assume Pa and need to provide a proof of $\forall z Qz$. We can prove the universal sentence $\forall z Qz$ by showing it is true for a specific but arbitrary constant.

The premise $\forall x (Pa \rightarrow Qx)$ gives us $Pa \rightarrow Qb$, which means that (with an assumption of Pa) we have a proof of Qb . Because b hasn't appeared in any undischarged assumptions, we can generalise this to the universal $\forall z Qz$. Discharging Pa , we can then prove $Pa \rightarrow \forall z Qz$.

It is important to note that here we must use both a and b as our arbitrary constants. The following proof would not work:

$$\frac{[Pa] \quad \frac{\frac{\forall x(Pa \rightarrow Qx)}{Pa \rightarrow Qa} \forall\text{Elim}}{\rightarrow\text{Elim}}}{\frac{Qa}{\forall zQz} \forall\text{Intro}} \rightarrow\text{Intro}$$

Here, we are not justified in applying $\forall\text{Intro}$. The constant a appears in the assumption Pa , which won't be discharged until the final step of the proof.

Universal 4

$$\frac{\frac{\frac{\forall xPx \wedge \forall yQy}{\forall xPx} \wedge\text{Elim} \quad \frac{\forall xPx \wedge \forall yQy}{\forall xQx} \wedge\text{Elim}}{\frac{Pa}{\forall xPx} \forall\text{Elim}} \quad \frac{\frac{\forall xQx}{Qa} \forall\text{Elim}}{\wedge\text{Intro}}}{\frac{Pa \wedge Qa}{\forall z(Pz \wedge Qz)} \forall\text{Intro}}$$

We need to prove a universal statement $\forall z(Pz \wedge Qz)$, which we can do by proving that the statement is true for a specific but arbitrary constant. In other words, we need to prove $Pa \wedge Qa$ without a appearing in any undischarged assumptions.

To prove $Pa \wedge Qa$, we need to provide a proof of Pa and a proof of Qa . We obtain Pa from $\forall xPx$, which we derive by $\wedge\text{Elim}$ from the premise $\forall xPx \wedge \forall yQy$. We obtain Qa from $\forall yQy$, which we also derive from the premise $\forall xPx \wedge \forall yQy$.

Universal 5

$$\frac{\frac{[\forall y_1 Py_1]}{Pa} \forall\text{Elim} \quad \frac{\forall x(Px \rightarrow Qx)}{Pa \rightarrow Qa} \forall\text{Elim}}{\rightarrow\text{Elim}} \quad \frac{Qa}{\forall y_2 Qy_2} \forall\text{Intro}}{\forall y_1 Py_1 \rightarrow \forall y_2 Qy_2} \rightarrow\text{Intro}$$

Our conclusion $\forall y_1 Py_1 \rightarrow \forall y_2 Qy_2$ is an implication statement. This means we can freely assume $\forall y_1 Py_1$ and we need to try to prove $\forall y_2 Qy_2$. To prove $\forall y_2 Qy_2$ we need to show it is true for an arbitrary constant. We need to try and prove Qa . Our assumption of $\forall y_1 Py_1$ gives us Pa , and our premise $\forall x(Px \rightarrow Qx)$ gives us $Pa \rightarrow Qa$. Together, these allow us to derive Qa .

Universal 6

$$\frac{\frac{\frac{\forall z(Pz \wedge Qz)}{Pa \wedge Qa} \forall\text{Elim} \quad \frac{\forall z(Pz \wedge Qz)}{Pa \wedge Qa} \forall\text{Elim}}{\frac{Pa}{\forall yPy} \wedge\text{Elim}} \wedge\text{Elim} \quad \frac{\frac{\forall z(Pz \wedge Qz)}{Pa \wedge Qa} \forall\text{Elim}}{\frac{Qa}{\forall yQy} \wedge\text{Elim}} \wedge\text{Elim}}{\frac{\forall yPy}{\forall yPy} \forall\text{Intro} \quad \frac{\forall yQy}{\forall yQy} \forall\text{Intro}} \forall\text{Intro} \quad \wedge\text{Intro}}{\forall yPy \wedge \forall yQy} \wedge\text{Intro}$$

Our conclusion is a conjunction of two sentences, so we need to provide two proofs: a proof of $\forall yPy$ and a proof of $\forall yQy$. On the left-hand side, we can derive $\forall yPy$ from a specific claim about an arbitrary constant, so we'll try to prove Pa . We can derive this from $Pa \wedge Qa$, which we can derive from the premise $\forall z(Pz \wedge Qz)$. The right-hand side works in a similar way: we use the premise to derive $Pa \wedge Qa$ and then Qa , which gives us $\forall yQy$.

Universal 7

$$\frac{[Pa] \quad \frac{\frac{\forall x(Px \rightarrow Qx)}{Pa \rightarrow Qa} \forall\text{Elim}}{Qa} \rightarrow\text{Elim} \quad \frac{\forall x\neg Qx}{\neg Qa} \forall\text{Elim}}{\frac{\neg Pa}{\forall x\neg Px} \forall\text{Intro}} \neg\text{Intro}$$

Our conclusion $\forall x\neg Px$ is a universal statement, so we can prove it by providing a proof of $\neg Pa$ without a appearing in any undischarged assumptions. To prove $\neg Pa$, we need to derive a contradiction from assumptions of Pa . Note that these assumptions of Pa will be discharged when we apply $\neg\text{Intro}$ and prove $\neg Pa$, so they won't interfere with our application of $\forall\text{Intro}$.

Where can we find the contradiction we need? One of our premises $\forall x\neg Qx$ gives us $\neg Qa$, so we have a contradiction if we can prove Qa . We do this by using our other premise $\forall x(Px \rightarrow Qx)$ to derive $Pa \rightarrow Qa$; this and our assumption of Pa allow us to prove Qa and obtain a contradiction.

Universal 8

$$\frac{\frac{\frac{[\forall x_1 Px_1]}{Pa} \forall\text{Elim} \quad \frac{[\forall x_2 Qx_2]}{Qa} \forall\text{Elim}}{\forall x_1 Px_1 \vee \forall x_2 Qx_2} \forall\text{Intro} \quad \frac{Pa \vee Qa}{Pa \vee Qa} \forall\text{Intro}}{\frac{Pa \vee Qa}{\forall x(Px \vee Qx)} \forall\text{Intro}} \forall\text{Elim}$$

In order to prove $\forall x(Px \vee Qx)$ we can prove $Pa \vee Qa$ is true without a appearing in any undischarged assumptions.

Our premise $\forall x_1 Px_1 \vee \forall x_2 Qx_2$ is a disjunction. We can apply $\forall\text{Elim}$ at the bottom of the proof to split the proof into two cases: one where we can freely assume $\forall x_1 Px_1$ and one where we can freely assume $\forall x_2 Qx_2$. On both sides we need to prove $Pa \vee Qa$.

On the left-hand side, our assumption of $\forall x_1 Px_1$ gives us Pa . From this, we can derive $Pa \vee Qa$ by applying \vee Intro. The right-hand side works in a similar way: $\forall x_2 Qx_2$ gives us Qa , then $Pa \vee Qa$.

Universal 9

$$\frac{\frac{\frac{[Pa]}{Pa \rightarrow Qb} \rightarrow\text{Elim} \quad \frac{\frac{\frac{\forall x \forall y (Px \rightarrow Qy)}{\forall y (Pa \rightarrow Qy)} \vee\text{Elim}}{\forall z Qz} \vee\text{Intro}}{Pa \rightarrow \forall z Qz} \rightarrow\text{Intro}}{\forall x (Px \rightarrow \forall z Qz)} \vee\text{Intro}}{Qb} \vee\text{Intro}}{\forall z Qz} \rightarrow\text{Intro}}{Pa \rightarrow \forall z Qz} \rightarrow\text{Intro}}{\forall x (Px \rightarrow \forall z Qz)} \vee\text{Intro}$$

Our conclusion $\forall x(Px \rightarrow \forall zQz)$ is a universal statement, so we can prove it by deriving $Pa \rightarrow \forall zQz$ as long as a doesn't appear in any undischarged assumptions. This is an implication, so we can prove it by assuming Pa and deriving $\forall zQz$. We can $\forall zQz$ from Qb , as long as b doesn't appear in any undischarged assumptions when we apply the \forall Intro step; note that we cannot use Qa for this because our assumption of Pa won't be discharged when we move to $\forall zQz$.

So how do we get from an assumption of Pa to Qb ? We use our premise $\forall x \forall y (Px \rightarrow Qy)$ to derive $\forall y (Pa \rightarrow Qy)$ and then $Pa \rightarrow Qb$. This means our assumption of Pa gives us the Qb we need.

Universal 10

$$\frac{\frac{\frac{[Pa]}{Pa \rightarrow Qa} \rightarrow\text{Elim} \quad \frac{\frac{\forall x (Px \rightarrow Qx)}{Qa} \vee\text{Elim}}{Qa} \rightarrow\text{Elim}}{Qa} \rightarrow\text{Elim} \quad \frac{\frac{\forall x (Qx \rightarrow Rx)}{Qa \rightarrow Ra} \vee\text{Elim}}{Qa \rightarrow Ra} \rightarrow\text{Elim}}{Ra} \rightarrow\text{Intro}}{Pa \rightarrow Ra} \rightarrow\text{Intro}}{\forall x (Px \rightarrow Rx)} \vee\text{Intro}$$

Our conclusion is a universal statement, so we need to prove it is true for an arbitrary instantiation. We can do this by proving $Pa \rightarrow Ra$ without a appearing in any undischarged assumptions. $Pa \rightarrow Ra$ is a conditional statement, so we can prove it by assuming Pa and trying to derive Ra .

We get from Pa to Ra with the help of our two premises. From $\forall x(Px \rightarrow Qx)$ we can derive $Pa \rightarrow Qa$, which (with our assumption of Pa) lets us prove Qa . From $\forall x(Qx \rightarrow Rx)$ we can derive $Qa \rightarrow Ra$, which lets us prove Ra . The application of \rightarrow Intro discharges Pa , leaving no undischarged assumptions containing a ; this means we're free to apply universal introduction and derive the conclusion.

Universal 11

$$\frac{\frac{\frac{\forall x(Px \vee Qx)}{Pa \vee Qa} \forall\text{Elim} \quad [Pa] \quad \frac{\frac{[\forall x \neg Qx]}{\neg Qa} \forall\text{Elim}}{Pa} \neg\text{Elim}}{\forall x Px} \forall\text{Intro}}{\neg \forall x \neg Qx} \neg\text{Intro}}{\neg \forall x \neg Qx} \neg\text{Intro}$$

Our conclusion $\neg \forall x \neg Qx$ is a negated statement, so we prove it by assuming $\forall x \neg Qx$ and deriving a contradiction. One of our premises is $\neg \forall x Px$, which is also a negated statement, so we will have the contradiction we need if we prove $\forall x Px$. A way of proving $\forall x Px$ is to prove Pa without a appearing in any undischarged assumptions.

Our other premise $\forall x(Px \vee Qx)$ is a disjunction, splitting the proof into a case where Pa is true and a case where Qa is true. In both cases we need to prove Pa ; in the former case this is trivial. In the latter case, proving Pa requires making use of our assumption $\forall x \neg Qx$. This gives us $\neg Qa$, which contradicts Qa and lets us derive Pa by $\neg\text{Elim}$.

Universal 12

$$\frac{\frac{\frac{\forall x(Px \wedge Qx)}{Pa \wedge Qa} \forall\text{Elim} \quad \frac{\forall x(Px \wedge Qx)}{Pb \wedge Qb} \forall\text{Elim}}{Pa} \wedge\text{Elim} \quad \frac{\frac{Pb \wedge Qb}{Qb} \wedge\text{Elim}}{Pa \wedge Qb} \wedge\text{Intro}}{\forall y(Pa \wedge Qy)} \forall\text{Intro}}{\forall x \forall y(Px \wedge Qy)} \forall\text{Intro}$$

Here our conclusion $\forall x \forall y(Px \wedge Qy)$ features two universal quantifiers, meaning we need to apply $\forall\text{Intro}$ twice. We can derive $\forall x \forall y(Px \wedge Qy)$ from $\forall y(Pa \wedge Qy)$ (provided a appears in no undischarged assumptions in the proof of $\forall y(Pa \wedge Qy)$) but we cannot derive $\forall y(Pa \wedge Qy)$ from $Pa \wedge Qa$. This is because the constant a appears in $\forall y(Pa \wedge Qy)$; this means we need to use two constants.

We will derive $\forall y(Pa \wedge Qy)$ from $Pa \wedge Qb$ and make sure that neither a nor b appear in any undischarged assumptions. Our premise $\forall x(Px \wedge Qx)$ gives us both $Pa \wedge Qa$ and $Pb \wedge Qb$, which allow us to obtain Pa and Qb by $\wedge\text{Elim}$.

Universal 13

$$\frac{\frac{\frac{\forall x \forall y Rxy}{\forall y Ray} \forall\text{Elim}}{Raa} \forall\text{Elim}}{\forall x Rxx} \forall\text{Intro}$$

Our conclusion $\forall x Rxx$ is a universal statement, so we can prove it by proving Raa without a appearing in any undischarged assumptions. We obtain Raa from our premise $\forall x \forall y Rxy$ and two applications of $\forall\text{Elim}$; in both case the variable is replaced by a .

Universal 14

$$\frac{\frac{\frac{[\forall x \forall y Rxy]}{\forall y Ray} \forall\text{Elim} \quad \frac{\forall x \neg \forall y Rxy}{\neg \forall y Ray} \forall\text{Elim}}{\neg \forall x \forall y Rxy} \neg\text{Intro}}$$

Our conclusion $\neg \forall x \forall y Rxy$ is a negated statement, so we prove it by assuming $\forall x \forall y Rxy$ and deriving a contradiction from it. Our premise $\forall x \neg \forall y Rxy$ isn't a negated statement itself, but we can apply $\forall\text{Elim}$ to obtain the negated statement $\neg \forall y Ray$ from it. Since we can obtain $\forall y Ray$ from our assumption of $\forall x \forall y Rxy$, we have the contradiction we need.

Universal 15

$$\frac{\frac{\frac{\forall x Rxx}{Raa} \forall\text{Elim} \quad \frac{[\forall y \neg Ray]}{\neg Raa} \forall\text{Elim}}{\neg \forall y \neg Ray} \neg\text{Intro}}{\forall x \neg \forall y \neg Ray} \forall\text{Intro}$$

Our conclusion $\forall x \neg \forall y \neg Ray$ is a universal statement, so we can derive it by proving $\neg \forall y \neg Ray$ without a appearing in any undischarged assumptions. This is a negated statement, so we prove it by assuming $\forall y \neg Ray$ and deriving a contradiction from it. Neither our premise $\forall x Rxx$ nor our assumption of $\forall y \neg Ray$ are negated statements, so we can't use them to obtain a contradiction immediately. However, from $\forall x Rxx$ we can derive Raa and from $\forall y \neg Ray$ we can derive $\neg Raa$, giving us the contradiction we need.

Universal 16

$$\frac{\frac{\frac{[\forall x \forall y (Rxy \vee Ryx)]}{\forall y (Ray \vee Rya)} \forall\text{Elim}}{Raa \vee Raa} \forall\text{Elim} \quad \frac{[Raa] \quad [Raa]}{Raa} \forall\text{Elim} \quad \frac{\forall x \neg Rxx}{\neg Raa} \forall\text{Elim}}{Raa \quad \neg Raa} \neg\text{Intro} \quad \neg \forall x \forall y (Rxy \vee Ryx)} \neg\text{Intro}$$

Our conclusion $\neg \forall x \forall y (Rxy \vee Ryx)$ is a negated statement, so we derive it by assuming $\forall x \forall y (Rxy \vee Ryx)$ and deriving a contradiction from it. Our premise $\forall x \neg Rxx$ isn't a negated statement we can use in a contradiction, but by applying $\forall\text{Elim}$ we can obtain $\neg Rxx$; this means we have a contradiction if we can somehow prove Raa .

From our assumption of $\forall x \forall y (Rxy \vee Ryx)$ we can apply $\forall\text{Elim}$ twice to obtain $Raa \rightarrow Raa$. This gives us Raa through a slightly bizarre application of $\forall\text{Elim}$ where Raa is trivially true in both cases. With Raa and $\neg Raa$ we have the contradiction we need to apply $\neg\text{Intro}$ and prove $\neg \forall x \forall y (Rxy \vee Ryx)$.

Universal 17

$$\frac{\frac{\frac{[\forall y Ryb]}{Rab} \forall\text{Elim} \quad \frac{[\forall y \neg Ray]}{\neg Rab} \forall\text{Elim}}{\neg \forall y Ryb} \neg\text{Intro}}{\forall x \neg \forall y Ryx} \forall\text{Intro} \quad \neg \forall x \neg \forall y Ryx}{\frac{\neg \forall y \neg Ray}{\forall x \neg \forall y \neg Rxy} \forall\text{Intro}} \neg\text{Intro}$$

Our conclusion $\forall x \neg \forall y \neg Rxy$ is a universal statement, so we will try to prove $\neg \forall y \neg Ray$ (making sure that a doesn't appear in any undischarged assumptions in the proof of $\neg \forall y \neg Ray$). $\neg \forall y \neg Ray$ is a negated statement, so we can prove it by assuming $\forall y \neg Ray$ and showing that it leads to a contradiction. Since our premise $\neg \forall x \neg \forall y Ryx$ is a negated statement, we will have the contradiction we need if we can prove $\forall x \neg \forall y Ryx$.

$\forall x \neg \forall y Ryx$ is a universal statement, so we can prove it by proving $\neg \forall y Ryb$ (as long as b doesn't appear in any undischarged assumptions in the proof of $\neg \forall y Ryb$). This is a negated statement, so we can prove it by assuming $\forall y Ryb$ and deriving a contradiction from it. $\forall y Ryb$ gives us Rab , and our other assumption of $\forall y \neg Ray$ gives us $\neg Rab$, so we have the contradiction we need.

Note that $\forall y Ryb$ is discharged before we apply the first $\forall\text{Intro}$ step, and $\forall y \neg Ray$ is discharged before we apply the second $\forall\text{Intro}$ step, so both applications are permitted.

Universal 18

$$\frac{\frac{\frac{\forall x Rxx}{Raa} \forall\text{Elim} \quad [Rab]}{Raa \wedge Rab} \wedge\text{Intro} \quad \frac{[\forall z \neg(Raz \wedge Rzb)]}{\neg(Raa \wedge Rab)} \forall\text{Elim}}{\frac{\neg \forall z \neg(Raz \wedge Rzb)}{Rab \rightarrow \neg \forall z \neg(Raz \wedge Rzb)} \rightarrow\text{Intro}}{\frac{\forall y(Ray \rightarrow \neg \forall z \neg(Raz \wedge Rzy))}{\forall x \forall y(Rxy \rightarrow \neg \forall z \neg(Rxz \wedge Rzy))} \forall\text{Intro}} \forall\text{Intro}$$

Our conclusion is a universal statement with two quantifiers, so we can prove it by proving $Rab \rightarrow \neg \forall z \neg(Raz \wedge Rzb)$, as long as a and b don't appear in any undischarged assumptions. This is a conditional statement, so we prove it by assuming Rab and deriving $\neg \forall z \neg(Raz \wedge Rzb)$. This in turn is a negated statement, so we can assume $\forall z \neg(Raz \wedge Rzb)$ and need to show it leads to a contradiction.

From $\forall z \neg(Raz \wedge Rzb)$ we can derive the negated statement $\neg(Raa \wedge Rab)$, so we have a contradiction if we can derive $Raa \wedge Rab$. We can obtain $Raa \wedge Rab$ from our premise (which gives us Raa) and our assumption of Rab . $\forall z \neg(Raz \wedge Rzb)$ and Rab are both discharged before the end of the proof, so we are free to apply the two \forall Intro steps.

We could also have carried out the proof in a slightly different way, using $\forall x Rxx$ to derive Rbb and using the assumption of $\forall z \neg(Raz \wedge Rzb)$ to derive $\neg(Rab \wedge Rbb)$. This would still have given us the contradiction we needed.

Universal 19

$$\frac{\frac{\frac{\forall x \forall y Rxy}{\forall y Ray} \forall\text{Elim} \quad \frac{\forall x \forall y Rxy}{\forall y Rby} \forall\text{Elim}}{\frac{Raa}{\forall y Rya} \forall\text{Elim}} \forall\text{Elim} \quad \frac{\frac{Rba}{\forall y Rya} \forall\text{Intro}}{\wedge\text{Intro}} \wedge\text{Intro}}{\frac{Raa \wedge \forall y Rya}{\forall x(Rxx \wedge \forall y Ryx)} \forall\text{Intro}} \forall\text{Intro}$$

Our conclusion $\forall x(Rxx \wedge \forall y Ryx)$ is a negated statement, so we can prove it by proving $Raa \wedge \forall y Rya$ without a appearing in any undischarged assumptions. $Raa \wedge \forall y Rya$ is a conjunction, so we need to provide a proof of Raa and a proof of $\forall y Rya$. On the left-hand side, we can derive Raa from our premise $\forall x \forall y Rxy$ by applying \forall Elim twice.

On the right-hand side, we need to prove the universal statement $\forall y Rya$, which we can prove by providing a proof of Rba without b appearing in any undischarged assumptions. Note that we cannot derive $\forall y Rya$ from Raa , even if a appears in no undischarged assumptions in the proof of Raa ; this is because the constant a occurs in $\forall y Rya$. We can prove Rba from the premise $\forall x \forall y Rxy$ by applying \forall Elim twice.

Universal 20

$$\frac{\frac{\frac{\forall x \forall y Rxy}{\forall y Ray} \forall\text{Elim}}{Rab} \forall\text{Elim}}{\frac{Rab \wedge Rba}{\forall y (Ray \wedge Rya)} \wedge\text{Intro}} \frac{\frac{\frac{\forall x \forall y Rxy}{\forall y Rby} \forall\text{Elim}}{Rba} \forall\text{Elim}}{\forall y (Ray \wedge Rya)} \wedge\text{Intro} \forall\text{Intro}$$

Our conclusion $\forall x \forall y (Rxy \wedge Ryx)$ is a universal statement, which we can derive from $\forall y (Ray \wedge Rya)$ (as long as a appears in no undischarged assumptions). This in turn we can derive from $Rab \wedge Rba$ (as long as b appears in no undischarged assumptions). Note that we would not be able to move from $Raa \wedge Raa$ to $\forall y (Ray \wedge Rya)$, because a still appears in $\forall y (Ray \wedge Rya)$; we have to use two separate constants.

To prove $Rab \wedge Rba$, we need to provide a proof of Rab and a proof of Rba . Both of these can be derived from our premise $\forall x \forall y Rxy$ through two applications of $\forall\text{Elim}$.

Universal 22

$$\frac{\frac{\frac{\forall x \forall y (\neg Qxy \vee \neg Qyx)}{\forall y (\neg Qay \vee \neg Qya)} \forall\text{Elim}}{\neg Qab \vee \neg Qba} \forall\text{Elim} \quad \frac{\frac{\frac{\frac{\forall x \forall y (Qxy \rightarrow Qyx)}{\forall y (Qay \rightarrow Qya)} \forall\text{Elim}}{Qab \rightarrow Qba} \forall\text{Elim}}{Qba} \rightarrow\text{Elim} \quad \frac{[Qab]}{\neg Qab} \rightarrow\text{Elim}}{[Qab]} \rightarrow\text{Elim} \quad \frac{[Qab]}{\neg Qba} \rightarrow\text{Intro}}{\neg Qab} \rightarrow\text{Intro}}{\frac{[Qab]}{\neg Qab} \rightarrow\text{Intro}} \rightarrow\text{Elim}}{\frac{\frac{[Qab]}{\neg Qab} \rightarrow\text{Intro}}{\forall y \neg Qay} \forall\text{Intro}}{\forall x \forall y \neg Qxy} \forall\text{Intro}} \rightarrow\text{Elim}} \forall\text{Elim}$$

Our conclusion $\forall x \forall y \neg Qxy$ features two universal quantifiers, so we can prove it by proving $\neg Qab$ and applying $\forall\text{Intro}$ twice. One of our premises $\forall x \forall y (\neg Qxy \vee \neg Qyx)$ gives us the disjunction $\neg Qab \vee \neg Qba$, splitting the proof into a case where we can assume $\neg Qab$ and a case where we can assume $\neg Qba$.

In the first case, $\neg Qab$ is exactly what we want to prove. In the other case, we can prove $\neg Qab$ by assuming Qab and showing it leads to a contradiction. Since the disjunction gives us an assumption of $\neg Qba$, it makes sense to try and prove Qba in order to give us the contradiction we need. We do this using our other premise $\forall x \forall y (Qxy \rightarrow Qyx)$, which gives us $Qab \rightarrow Qba$.

Universal 23

$$\begin{array}{c}
 \frac{\frac{[Rab]^2 \quad [Rbc]^1}{Rab \wedge Rbc} \wedge\text{Intro} \quad \frac{\frac{[\forall y \forall z \neg (Ray \wedge Ryz)]^3}{\forall z \neg (Rab \wedge Rbz)} \forall\text{Elim}}{\neg (Rab \wedge Rbc)} \forall\text{Elim}}{\frac{\neg Rbc}{\forall y \neg Rby} \forall\text{Intro}} \neg\text{Intro}^1 \quad \frac{\forall x \neg \forall y \neg Rxy}{\neg \forall y \neg Rby} \forall\text{Elim}}{\frac{\neg Rab}{\forall y \neg Ray} \forall\text{Intro}} \neg\text{Intro}^2 \quad \frac{\forall x \neg \forall y \neg Rxy}{\neg \forall y \neg Ray} \forall\text{Elim}}{\frac{\neg \forall y \forall z \neg (Ray \wedge Ryz)}{\forall x \neg \forall y \forall z \neg (Rxy \wedge Ryz)} \forall\text{Intro}} \neg\text{Intro}^3
 \end{array}$$

Our conclusion $\forall x \neg \forall y \forall z \neg (Rxy \wedge Ryz)$ is a universal statement, so we prove it by proving $\neg \forall y \forall z \neg (Ray \wedge Ryz)$ without a appearing in any undischarged assumptions. This is a negated statement, so we need to derive a contradiction from an assumption of $\forall y \forall z \neg (Ray \wedge Ryz)$. Since our premise $\forall x \neg \forall y \neg Rxy$ gives us $\neg \forall y \neg Ray$, we have the contradiction we need if we can prove $\forall y \neg Ray$.

We can derive $\forall y \neg Ray$ from $\neg Rab$ as long as b doesn't appear in any undischarged assumptions. Proving $\neg Rab$ requires assuming Rab and deriving another contradiction. To get this second contradiction, we need to use our premise again, but this time we derive $\neg \forall y \neg Rby$. We have a contradiction if we can prove $\forall y \neg Rby$.

This in turn can be derived from $\neg Rbc$, as long as c doesn't appear in any undischarged assumptions. $\neg Rbc$ can be proved by assuming Rbc and showing that it leads to a contradiction.

We've now made three assumptions: Rab , Rbc and $\forall y \forall z \neg (Ray \wedge Ryz)$. From the latter we can derive $\neg (Rab \wedge Rbc)$, which gives us the first contradiction we need. Because this contradiction discharges Rbc , c is left in no undischarged assumptions and we can derive $\forall y \neg Rby$. Because the second contradiction discharges Rab , b is left in no undischarged assumptions and we are free to derive $\forall y \neg Ray$. Because the final contradiction discharges $\forall y \forall z \neg (Ray \wedge Ryz)$, a is left in no undischarged assumptions and we can derive our conclusion.

Universal 25

$$\begin{array}{c}
 \frac{\frac{\frac{[(Rab \wedge Rbc) \wedge Rca]}{Rab \wedge Rbc} \wedge E}{Rac} \rightarrow E}{Rac \wedge Rca} \rightarrow E \quad \frac{\frac{\frac{\frac{\frac{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxx)}{\forall y \forall z ((Ray \wedge Ryz) \rightarrow Raz)} \forall E}{\forall z ((Rab \wedge Rbz) \rightarrow Raz)} \forall E}{(Rab \wedge Rbc) \rightarrow Rac} \forall E}{Rca} \wedge I}{[(Rab \wedge Rbc) \wedge Rca] \wedge E} \wedge E \quad \frac{\frac{\frac{\frac{\frac{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxx)}{\forall y \forall z ((Ray \wedge Ryz) \rightarrow Raz)} \forall E}{\forall z ((Rac \wedge Rcz) \rightarrow Raz)} \forall E}{(Rac \wedge Rca) \rightarrow Raa} \rightarrow E}{Raa} \rightarrow E \quad \frac{\frac{\forall x \neg Rxx}{\neg Raa} \forall E}{\neg Raa} \forall E}{\neg((Rab \wedge Rbc) \wedge Rca)} \forall I}{\forall z \neg((Rab \wedge Rbz) \wedge Rza)} \forall I}{\forall y \forall z \neg((Ray \wedge Ryz) \wedge Rza)} \forall I}{\forall x \forall y \forall z \neg((Rxy \wedge Ryz) \wedge Rzx)} \forall I} \neg I
 \end{array}$$

84

This is adapted from a past paper question from 2014. In order to prove $\forall x \forall y \forall z \neg((Rxy \wedge Ryz) \wedge Rzx)$ we can prove $\neg((Rab \wedge Rbc) \wedge Rca)$ without a, b or c appearing in any undischarged assumptions; this is a negated statement, so we need to derive a contradiction from assumptions of $(Rab \wedge Rbc) \wedge Rca$. Our premise $\forall x \neg Rxx$ could give us $\neg Raa$, $\neg Rbb$ and $\neg Rcc$, so we have a contradiction if we can prove Raa , Rbb or Rcc .

Although it is possible to prove any of them, Raa turns out to be the easiest to prove. From our assumption $(Rab \wedge Rbc) \wedge Rca$ we can derive $Rab \wedge Rbc$; we can use this with our premise to derive Rac . Our assumption provides Rca , so with both of them we can derive $Rac \wedge Rca$. In order to go from $Rac \wedge Rca$ to Raa , we need to use our premise to derive $(Rac \wedge Rca) \rightarrow Raa$. Raa gives us the contradiction we need to discharge $(Rab \wedge Rbc) \wedge Rca$ and derive $\neg((Rab \wedge Rbc) \wedge Rca)$.

Universal 26

$$\begin{array}{c}
 \frac{\frac{[Rab \wedge Rbc]}{Rab} \wedge E \quad \frac{\forall x Rxx}{Raa} \forall E}{Rab \wedge Raa} \wedge I \quad \frac{\frac{\frac{\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz)}{\forall y \forall z ((Ray \wedge Raz) \rightarrow Ryz)} \forall E}{\forall z ((Rab \wedge Raz) \rightarrow Rbz)} \forall E}{(Rab \wedge Raa) \rightarrow Rba} \forall E}{Rba} \rightarrow E \\
 \frac{Rba \wedge Rbc}{Rba} \wedge I \quad \frac{[Rab \wedge Rbc]}{Rbc} \wedge E \quad \frac{\frac{\frac{\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz)}{\forall y \forall z ((Rby \wedge Rbz) \rightarrow Ryz)} \forall E}{\forall z ((Rba \wedge Rbz) \rightarrow Raz)} \forall E}{(Rba \wedge Rbc) \rightarrow Rac} \forall E}{Rac} \rightarrow E \\
 \frac{Rac}{(Rab \wedge Rbc) \rightarrow Rac} \rightarrow I \\
 \frac{\frac{\frac{\frac{\frac{\frac{\forall z ((Rab \wedge Rbz) \rightarrow Raz)}{\forall y \forall z ((Ray \wedge Ryz) \rightarrow Raz)} \forall I}{\forall y \forall z ((Ray \wedge Ryz) \rightarrow Raz)} \forall I}{\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)} \forall I}}{\forall z ((Rab \wedge Rbz) \rightarrow Raz)} \forall I}{(Rab \wedge Rbc) \rightarrow Rac} \forall I \\
 \frac{(Rab \wedge Rbc) \rightarrow Rac}{Rac} \rightarrow E
 \end{array}$$

85

This is adapted from a past paper question from 2011. In order to prove $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$ we can prove $(Rab \wedge Rbc) \rightarrow Rac$, as long as a , b and c don't appear in any undischarged assumptions. This is an implication, so we can assume $Rab \wedge Rbc$ and need to derive Rac . This much is straightforward; more tricky is working out how to use our premises to get from $Rab \wedge Rbc$ to Rac . $\forall x Rxx$ allows us to derive Raa , Rbb or Rcc if we want them; $\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz)$ gives us many different sentences involving a , b and c . We need to work out which sentences we need.

Rac is the sentence we're aiming for, so ideally we want to use our premise $\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow Ryz)$ to derive a conditional with Rac at the end. We could derive $(Raa \wedge Rac) \rightarrow Rac$, but this wouldn't help us very much: in order to use it we would need a proof of Rac , and a proof of Rac is what we're looking for in the first place. We could also try $(Rca \wedge Rcc) \rightarrow Rac$, and this would work, but would lead to a proof which is much longer than necessary. So instead we will use all three of our constants and derive $(Rba \wedge Rbc) \rightarrow Rac$.

This means we need a proof of $Rba \wedge Rbc$, which requires a proof of Rba and a proof of Rbc . Rbc is easy; it comes directly from our assumption. Proving Rba requires using our big premise to derive another conditional, this time with Rba as the consequent. $(Rab \wedge Raa) \rightarrow Rba$ is the ideal conditional to derive, because we have both Rab (from our assumption) and Raa (from our premise). This means we have a complete proof.

5.8 Existential quantifier

Existential 1

$$\frac{\frac{\exists xPx \quad \frac{[Pa]}{\exists yPy} \exists\text{Intro}}{\exists yPy} \exists\text{Elim}}{\exists yPy} \exists\text{Elim}}$$

This proof involves one use of each rule for the existential quantifier. Although there are only two steps, this proof exemplifies the unusual structure which proofs involving the existential quantifier take: we eliminate any existential quantifiers at the bottom, and introduce them at the top. This means there are two ways we can look at this proof: one from top to bottom and one from bottom to top.

From top to bottom, we start by assuming Pa , and then apply $\exists\text{Intro}$ to derive $\exists yPy$. Following this we make use of our premise $\exists xPx$ and apply $\exists\text{Elim}$. This allows us to discharge our assumptions of Pa , provided that a doesn't appear in any other undischarged assumptions, and doesn't appear in either $\exists yPy$ or $\exists xPx$. All of these conditions are satisfied, so we are allowed to apply $\exists\text{Elim}$ and discharge Pa .

From bottom to top, we can see our premise $\exists xPx$ as giving us a 'free' assumption of Pa , provided that a is an arbitrary constant. a can't appear in $\exists xPx$ or in what we ultimately plan to prove. It also can't appear in any other undischarged assumptions we use. It is this assumption of Pa which we use to apply $\exists\text{Intro}$ and derive our conclusion, $\exists yPy$.

Existential 2

$$\frac{\frac{\frac{[Pa]}{\exists xPx} \exists\text{Intro} \quad \neg\exists xPx}{\neg Pa} \neg\text{Intro}}{\exists x\neg Px} \exists\text{Intro}}{\exists x\neg Px} \exists\text{Intro}}$$

Our conclusion is an existential statement, so we can prove it from $\neg Pa$ by applying $\exists\text{Intro}$. $\neg Pa$ is a negation, so we prove it by assuming Pa and showing that it leads to a contradiction. Our premise $\neg\exists xPx$ is a negated statement, so we have a contradiction if we can prove $\exists xPx$. We can derive this from our assumption of Pa by applying $\exists\text{Intro}$, giving us the contradiction we need.

We could have used any constant in place of a in this proof, because the two $\exists\text{Intro}$ steps would still have worked. Although our choice of a here was arbitrary, in other proofs we might not have so much flexibility.

Existential 3

$$\frac{\frac{\frac{[Pa] \quad [Pa \rightarrow Qb]}{\rightarrow\text{Elim}}}{Qb} \quad \frac{Qb}{\exists x_2 Qx_2} \exists\text{Intro}}{\exists x_1(Pa \rightarrow Qx_1) \quad Pa \rightarrow \exists x_2 Qx_2} \rightarrow\text{Intro}}{Pa \rightarrow \exists x_2 Qx_2} \exists\text{Elim}$$

Our premise $\exists x_1(Pa \rightarrow Qx_1)$ gives us a free assumption of $Pa \rightarrow Qb$ which we can use to derive our conclusion $Pa \rightarrow \exists x_2 Qx_2$. Note that I've chosen to use b as our arbitrary constant because we can't use a : a can't be our arbitrary constant because it appears in $\exists x_1(Pa \rightarrow Qx_1)$.

Our conclusion is a conditional, so we can assume Pa and need to try to derive $\exists x_2 Qx_2$. With our assumptions of Pa and $Pa \rightarrow Qb$ we can derive Qb , which means we can derive $\exists x_2 Qx_2$ by $\exists\text{Intro}$.

In the proof above we apply $\exists\text{Elim}$ at the end of the proof to discharge $Pa \rightarrow Qb$; we could also have applied it before applying $\rightarrow\text{Intro}$, as shown in the proof below. Usually it is most straightforward to wait until the very end of the proof before applying $\exists\text{Elim}$, but in later examples we'll see cases where this isn't possible.

$$\frac{\frac{\frac{[Pa] \quad [Pa \rightarrow Qb]}{\rightarrow\text{Elim}}}{Qb} \quad \frac{Qb}{\exists x_2 Qx_2} \exists\text{Intro}}{\exists x_1(Pa \rightarrow Qx_1) \quad \exists x_2 Qx_2} \exists\text{Elim}}{\frac{\exists x_2 Qx_2}{Pa \rightarrow \exists x_2 Qx_2} \rightarrow\text{Intro}}$$

Existential 4

$$\frac{\frac{\frac{[Pa \wedge Qa]}{Pa} \wedge\text{Elim} \quad \frac{[Pa \wedge Qa]}{Qa} \wedge\text{Elim}}{\exists y Py} \exists\text{Intro} \quad \frac{Qa}{\exists z Qz} \exists\text{Intro}}{\exists x(Px \wedge Qx) \quad \exists y Py \wedge \exists z Qz} \wedge\text{Intro}}{\exists y Py \wedge \exists z Qz} \exists\text{Elim}$$

Our premise $\exists x(Px \wedge Qx)$ is an existential statement, which gives us a free assumption of $Pa \wedge Qa$. We need to derive $\exists y Py \wedge \exists z Qz$, so we need to provide a proof of $\exists y Py$ and a proof of $\exists z Qz$. On the left, our assumption of $Pa \wedge Qa$ gives us Pa , which gives us $\exists y Py$; on the right, our assumption gives us Qa , which gives us $\exists z Qz$. When we apply $\exists\text{Elim}$, both assumptions of $Pa \wedge Qa$ are discharged at once, which the rules governing $\exists\text{Elim}$ do allow.

Again it's possible for us to apply $\exists\text{Elim}$ sooner than the end of the proof. This results in a proof which is slightly longer, as shown below:

$$\frac{\frac{\frac{[Pa \wedge Qa]}{Pa} \wedge\text{Elim}}{\exists x(Px \wedge Qx)} \exists\text{Intro}}{\exists yPy} \exists\text{Elim} \quad \frac{\frac{[Pa \wedge Qa]}{Qa} \wedge\text{Elim}}{\exists zQz} \exists\text{Intro}}{\exists yPy} \exists\text{Elim}}{\exists yPy \wedge \exists zQz} \wedge\text{Intro}$$

In this proof we apply $\exists\text{Elim}$ twice, and on both occasions we use a as our arbitrary constant. This is allowed, but isn't necessary: we could use a different constant in each branch. In certain proofs we apply $\exists\text{Elim}$ twice in the same branch, so that one application follows another; in these proofs we are obliged to use different constants each time.

Existential 5

$$\frac{\frac{\frac{[Pa]}{\exists yPy} \exists\text{Intro}}{[Pa \vee Qa]} \vee\text{Intro} \quad \frac{\frac{[Qa]}{\exists zQz} \exists\text{Intro}}{\exists yPy \vee \exists zQz} \vee\text{Intro}}{\exists yPy \vee \exists zQz} \vee\text{Elim}}{\exists x(Px \vee Qx)} \exists\text{Elim}$$

Our premise is an existential statement, giving us a free assumption of $Pa \vee Qa$. This is a disjunction, splitting the proof into a case where Pa is true and a case where Qa is true. In the case where Pa is true we can derive $\exists yPy$ and hence our conclusion $\exists yPy \vee \exists zQz$; in the other case we can derive $\exists yPy$ and can also derive the conclusion $\exists yPy \vee \exists zQz$. Our assumptions of Pa and Qa are discharged when we apply $\vee\text{Elim}$, so they do not interfere with our application of $\exists\text{Elim}$.

Existential 6

$$\frac{\frac{\frac{[Pa]}{Pa \vee Qa} \vee\text{Intro}}{[\exists xPx]} \exists\text{Intro} \quad \frac{\frac{[Qa]}{Pa \vee Qa} \vee\text{Intro}}{[\exists yQy]} \exists\text{Intro}}{\exists z(Pz \vee Qz)} \exists\text{Elim} \quad \frac{[\exists xPx]}{\exists z(Pz \vee Qz)} \exists\text{Elim} \quad \frac{[\exists yQy]}{\exists z(Pz \vee Qz)} \exists\text{Elim}}{\exists z(Pz \vee Qz)} \vee\text{Elim}$$

Our premise is a disjunction, splitting the proof into a case where $\exists xPx$ is true (on the left) and a case where $\exists yQy$ is true (on the right). On the left-hand side we use our assumption of $\exists xPx$ to discharge another assumption of Pa ; this gives us $Pa \vee Qa$ and hence our conclusion $\exists z(Pz \vee Qz)$. The right-hand side works similarly: $\exists yQy$ gives us a free assumption of Qa , from which we can derive $Pa \vee Qa$ and $\exists z(Pz \vee Qz)$.

Existential 7

$$\begin{array}{c}
 \frac{[Pa]^2 \quad Pa \rightarrow \exists xQx}{\exists xQx} \rightarrow\text{Elim} \quad \frac{\frac{[Qb]^1}{Pa \rightarrow Qb} \rightarrow\text{Intro}}{\exists x(Pa \rightarrow Qx)} \exists\text{Intro}}{\exists x(Pa \rightarrow Qx)} \exists\text{Elim}^1 \quad \frac{[\neg\exists x(Pa \rightarrow Qx)]^3}{\neg Pa} \neg\text{Intro}^2 \quad [Pa] \neg\text{Elim}}{\frac{Qc}{Pa \rightarrow Qc} \rightarrow\text{Intro}} \neg\text{Elim} \\
 \frac{\frac{Qc}{Pa \rightarrow Qc} \rightarrow\text{Intro}}{\exists x(Pa \rightarrow Qx)} \exists\text{Intro} \quad \frac{[\neg\exists x(Pa \rightarrow Qx)]^3}{\exists x(Pa \rightarrow Qx)} \neg\text{Elim}^3
 \end{array}$$

This proof is nastier than it might look at first. Our premise $Pa \rightarrow \exists xQx$ isn't much help to us on its own, but with an assumption of Pa it allows us to derive $\exists xQx$. This gives us a free assumption of Qb which we can use to derive $\exists x(Pa \rightarrow Qx)$; we can't assume Qa because it appears in $\exists x(Pa \rightarrow Qx)$ and Pa (which won't be discharged by the time we apply $\exists\text{Elim}$).

From Qb we can straightforwardly derive $Pa \rightarrow Qb$ and then $\exists x(Pa \rightarrow Qx)$, and we can then apply $\exists\text{Elim}$ to discharge Qb . But there is a problem: our assumption of Pa still hasn't been discharged, so we don't have a complete proof of $\exists x(Pa \rightarrow Qx)$.

It turns out that this is an indirect proof: we need to assume $\neg\exists x(Pa \rightarrow Qx)$ and show that this leads to a contradiction. Our first assumption of $\neg\exists x(Pa \rightarrow Qx)$ lets us discharge Pa and derive $\neg Pa$; from $\neg Pa$ we can derive $\exists x(Pa \rightarrow Qx)$ again.

One way of deriving $\exists x(Pa \rightarrow Qx)$ is to assume Pa , derive Qc in one step by $\neg\text{Elim}$, and apply $\rightarrow\text{Intro}$ to discharge Pa and derive $Pa \rightarrow Qc$. $\exists x(Pa \rightarrow Qx)$ again contradicts $\neg\exists x(Pa \rightarrow Qx)$, so we apply $\neg\text{Elim}$, discharge our assumptions of $\neg\exists x(Pa \rightarrow Qx)$ and derive $\exists x(Pa \rightarrow Qx)$.

Existential 8

$$\frac{\frac{\frac{[Raa]}{Raa \rightarrow Raa} \rightarrow\text{Intro}}{\exists y(Ray \rightarrow Rya)} \exists\text{Intro}}{\exists x \exists y(Rxy \rightarrow Ryx)} \exists\text{Intro}}$$

The statement we want to prove is $\exists x \exists y(Rxy \rightarrow Ryx)$. We can apply $\exists\text{Intro}$ twice to derive this from a number of different statements, but not all of them will be useful. For example, we could derive $\exists x \exists y(Rxy \rightarrow Ryx)$ from $Rab \rightarrow Rba$, but we have no way of providing a proof of $Rab \rightarrow Rba$ from no premises.

Instead, we will derive $\exists x \exists y(Rxy \rightarrow Ryx)$ from $Raa \rightarrow Raa$. We can prove $Raa \rightarrow Raa$ from no premises by assuming Raa and applying implication-introduction, discharging the assumption.

Note that the rules of Natural Deduction do allow us to make the first $\exists\text{Intro}$ step and only replace some of the as with ys . This is a difference between $\exists\text{Intro}$ and $\forall\text{Intro}$: when applying the latter rule, all occurrences of the constant need to be replaced with the variable.

Existential 9

$$\frac{\frac{\frac{[Rab]^1}{\exists y Ryb} \exists\text{Intro}}{[\exists y Ray]^2} \exists\text{Intro}}{\exists x \exists y Rxy} \exists\text{Intro}}{\frac{\frac{\frac{[\exists y Ray]^2}{\exists x \exists y Rxy} \exists\text{Intro}}{\exists x \exists y Ryx} \exists\text{Elim}^1}}{\exists x \exists y Ryx} \exists\text{Elim}^2}}$$

In this proof our premise $\exists x \exists y Rxy$ contains two existential quantifiers. This means we need to apply $\exists\text{Elim}$ twice. The way we do this is quite mechanical, but produces an odd-looking proof structure.

Applying $\exists\text{Elim}$ with $\exists x \exists y Rxy$ at the bottom of the proof allows us to discharge an assumption of $\exists y Ray$. Applying $\exists\text{Elim}$ with $\exists y Ray$ allows us to discharge an assumption of Rab . Using this assumption, we apply $\exists\text{Intro}$ twice to derive our conclusion $\exists x \exists y Ryx$.

It's worth verifying that our $\exists\text{Elim}$ steps are allowed. The first application of $\exists\text{Elim}$ (higher up in the proof) replaces y with b ; it is allowed because b doesn't appear in $\exists y Ray$, $\exists x \exists y Ryx$ or in any undischarged assumptions in the proof of $\exists x \exists y Ryx$ other than Rab , which is then discharged. Our second application (at the bottom of the proof) replaces x with a ; it is allowed because a doesn't appear in $\exists x \exists y Rxy$, $\exists x \exists y Ryx$ or in any undischarged assumptions other than $\exists y Ray$, which is then discharged.

If we had used only one constant throughout the proof, the proof would not have worked. An example of an incorrect proof is shown below:

$$\frac{\frac{\frac{[Raa]^1}{\exists y Rya} \exists \text{Intro}}{\exists x \exists y Rxy} \exists \text{Intro}}{\exists x \exists y Ryx} \exists \text{Elim}^1}{\exists x \exists y Ryx} \exists \text{Elim}^2$$

Here, the first application of $\exists \text{Elim}$ (attempting to discharge Raa) is not allowed, because the constant a appears in $\exists y Rya$.

Existential 10

$$\frac{\frac{\frac{[Raa] \quad [Raa]}{Raa \wedge Raa} \wedge \text{Intro}}{\exists y (Ray \wedge Rya)} \exists \text{Intro}}{\exists x Rxx \quad \exists x \exists y (Rxy \wedge Ryx)} \exists \text{Intro}}{\exists x \exists y (Rxy \wedge Ryx)} \exists \text{Elim}$$

Our premise $\exists x Rxx$ allows us to discharge an assumption of Raa . This is useful because we can use it twice to derive $Raa \wedge Raa$, which allows us to derive $\exists x \exists y (Rxy \wedge Ryx)$ by $\exists \text{Intro}$.

Existential 11

$$\frac{\frac{\frac{[Raa]}{\exists y Ray} \exists \text{Intro}}{\exists x \exists y Rxy} \exists \text{Intro}}{\exists x \exists y Rxy} \exists \text{Elim} \quad \neg \exists x \exists y Rxy}{\neg \exists y Ryy} \neg \text{Intro}$$

Here our conclusion is a negated statement, meaning we need to derive a contradiction from an assumption of $\exists y Ryy$. Our premise $\neg \exists x \exists y Rxy$ is a negated statement, so we have a contradiction if we can derive $\exists x \exists y Rxy$. Applying $\exists \text{Elim}$ to this assumption of $\exists y Ryy$ gives us an assumption of Raa . From this assumption we can derive $\exists x \exists y Rxy$, giving us the contradiction we need.

Unusually, we don't apply $\exists \text{Elim}$ at the very end of the proof. When we apply $\neg \text{Intro}$ we need to discharge our assumption of $\exists y Ryy$, so we need to apply $\exists \text{Elim}$ before we apply $\neg \text{Intro}$.

Existential 13

$$\frac{\frac{\frac{[\exists x Rxx]^1}{\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)} \vee I \quad [\neg(\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx))]^3}{\neg \exists x Rxx} \neg I^1 \quad \frac{[Raa]^2}{\exists x Rxx} \exists I}{\exists x Rxx} \neg E}{\frac{\frac{\frac{\frac{\neg \exists y Rya}{Raa \rightarrow \neg \exists y Rya} \rightarrow I^2}{\exists x (Rxx \rightarrow \neg \exists y Ryx)} \exists I}{\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)} \vee I \quad [\neg(\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx))]^3}{\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)} \neg E^3}$$

What we want to prove is a disjunction, but we can't provide a direct proof of either disjunct. This turns out to be an indirect proof, of a very similar form to the proof of $P \vee (P \rightarrow Q)$ given in the negation section. We have to assume $\neg \exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)$, the negation of our conclusion, and show that it leads to a contradiction.

We start by assuming $\exists x Rxx$, from which we can derive the conclusion $\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)$. Assuming $\neg(\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx))$ gives us a contradiction, which lets us discharge $\exists x Rxx$ and derive $\neg \exists x Rxx$.

We do the actual legwork of the proof when we derive $\exists x (Rxx \rightarrow \neg \exists y Ryx)$ from $\neg \exists x Rxx$. We can derive this from $Raa \rightarrow \neg \exists y Rya$, which is an implication: so we assume Raa and derive $\neg \exists y Rya$.

This assumption of Raa allows us to derive $\exists x Rxx$, which contradicts $\neg \exists x Rxx$ and lets us derive $\neg \exists y Rya$ by \neg Elim. This means we can apply implication-introduction (discharging Raa), \exists Intro and \vee Intro to prove $\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)$.

Finally we assume $\neg(\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx))$ a second time, letting us discharge both occurrences of $\neg(\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx))$ and derive $\exists x Rxx \vee \exists x (Rxx \rightarrow \neg \exists y Ryx)$ by \neg Elim.

The above is not the only way in which this proof could have been carried out. We could have begun, for example, by assuming Raa . It would also be possible to carry out the proof by first assuming $\exists x (Rxx \rightarrow \neg \exists y Ryx)$, or by first assuming $\neg Raa$.

Existential 14

$$\begin{array}{c}
 \frac{\frac{\frac{\neg Qa \quad [Qb]}{\neg Qa \wedge Qb} \wedge I \quad \frac{Rab \wedge Rbc}{Rab} \wedge E}{(\neg Qa \wedge Qb) \wedge Rab} \wedge I}{\frac{\exists y((\neg Qa \wedge Qy) \wedge Ray)}{\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)} \exists I} \exists I \quad \frac{[\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)]}{\neg Qb} \neg I \\
 \frac{\frac{\frac{\frac{\neg Qb \wedge Qc}{(\neg Qb \wedge Qc) \wedge Rbc} \wedge I \quad \frac{Qc \quad \frac{Rab \wedge Rbc}{Rbc} \wedge E}{Rbc} \wedge I}{\exists y((\neg Qb \wedge Qy) \wedge Rby)} \exists I}{\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)} \exists I \quad \frac{[\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)]}{\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)} \neg E
 \end{array}$$

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This is adapted from a past paper question from 2016. The good news is our premises have no quantifiers, but the bad news is that we have no way of proving our conclusion $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ directly from these premises.

What we will do first is assume Qb and derive the conclusion $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ from this. Then we will assume $\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ and apply \neg -Intro to discharge Qb and derive $\neg Qb$. We will then show that even with $\neg Qb$ we can derive $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$. At this point we will assume $\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ again and apply \neg -Elim, discharging both assumptions of $\neg \exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ and providing a proof of $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$.

The proofs of $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$ from Qb and from $\neg Qb$ are fairly straightforward, and similar to each other. In the first proof we aim to prove $(\neg Qa \wedge Qb) \wedge Rab$, which we can do easily with the conjunction rules, our premises and our assumption of Qb . In the second proof we aim to prove $(\neg Qb \wedge Qc) \wedge Rbc$, which we can do with our premises and proof of $\neg Qb$. Then both times we apply \exists Intro twice to arrive at $\exists x \exists y((\neg Qx \wedge Qy) \wedge Rxy)$.

Existential 15

$$\frac{\frac{\frac{\frac{[Pabc]^1}{\exists zPazc} \exists\text{Intro}}{\exists y\exists zPyzc} \exists\text{Intro}}{\exists x\neg\exists y\exists z\neg Pyzx]^3} \exists z\neg Pbz} \frac{[\exists y\neg\exists z\neg Payz]^5}{\exists x\exists y\neg\exists z\neg Pxyz} \frac{\frac{\frac{\frac{\frac{\frac{[\neg\exists y\exists zPyzc]^2}{\neg\exists z\neg Pabz]^4} \neg\text{Intro}^3}{\exists z\neg Pbz} \exists\text{Intro}}{\exists z\neg Pbz} \exists\text{Elim}^2} \neg\exists x\neg\exists y\exists zPyzx} \exists\text{Elim}^4} \neg\exists x\neg\exists y\exists zPyzx} \exists\text{Elim}^5} \exists\text{Elim}^1} \exists\text{Elim}^2} \exists\text{Elim}^3} \exists\text{Elim}^4} \exists\text{Elim}^5} \exists\text{Elim}^6} \exists\text{Elim}^7} \exists\text{Elim}^8} \exists\text{Elim}^9} \exists\text{Elim}^{10}$$

This proof involves ternary predicates, lots of discharging of assumptions and three applications of $\exists\text{Elim}$. The good news is that the proof is quite systematic.

With our premise $\exists x\exists y\neg\exists z\neg Pxyz$ we can apply $\exists\text{Elim}$ at the end of the proof to discharge assumptions of $\exists y\neg\exists z\neg Payz$, as long as a doesn't appear in any other undischarged assumptions by that stage. $\exists y\neg\exists z\neg Payz$ is itself an existential statement, so we can use it to discharge assumptions of $\neg\exists z\neg Pabz$, as long as b doesn't appear in any undischarged assumptions by the time we make this application of $\exists\text{Elim}$.

What we want to prove is $\neg\exists x\neg\exists y\exists zPyzx$, a negation, so we need to assume $\exists x\neg\exists y\exists zPyzx$ and show that this leads to a contradiction. The assumption which our premise gives us is $\neg\exists z\neg Pabz$, so we can obtain a contradiction by deriving $\exists z\neg Pabz$.

Assuming $\exists x\neg\exists y\exists zPyzx$ (which we have assumed for our first contradiction), we can apply $\exists\text{Elim}$ to discharge assumptions of $\neg\exists y\exists zPyzc$, as long as c doesn't appear in any undischarged assumptions at this point. From this we need to derive $\exists z\neg Pabz$, which can be obtained from $\neg Pabc$. $\neg Pabc$ is itself a negated statement, so we obtain it by assuming $Pabc$ and deriving a second contradiction. To do this derive $\exists y\exists zPyzc$ from $Pabc$, contradicting our assumption of $\neg\exists y\exists zPyzc$.

Existential 16

$$\frac{\frac{[Pa]}{\exists x Px} \exists\text{Intro} \quad \neg\exists x Px}{\neg Pa} \neg\text{Intro}}{\forall x \neg Px} \forall\text{Intro}$$

This is the first of four proofs illustrating the duality between the universal quantifier and the existential quantifier. The proofs aren't very intuitive, but the techniques appear in lots of harder proofs.

We can prove $\forall x \neg Px$ by proving $\neg Pa$ as long as Pa doesn't appear in any undischarged assumptions. This is a negation, so we can derive it by assuming Pa and deriving a contradiction. Our premise $\neg\exists x Px$ is also a negated statement, so we have a contradiction if we can prove $\exists x Px$. This follows from our assumption of Pa by $\exists\text{Intro}$, so we have the contradiction we need.

Because our application of $\neg\text{Intro}$ discharges our assumption of Pa , we are free to apply $\forall\text{Intro}$ in the last step and derive the conclusion $\forall x \neg Px$.

Existential 17

$$\frac{\frac{\frac{[\forall x Px]}{Pa} \forall\text{Elim} \quad [\neg Pa]}{\neg\forall x Px} \neg\text{Intro}}{\exists x \neg Px} \exists\text{Elim}}{\neg\forall x Px} \exists\text{Elim}$$

Our premise $\exists x \neg Px$ is an existential statement, so by applying $\exists\text{Elim}$ at the end of the proof we can discharge an assumption of $\neg Pa$ (as long as a doesn't appear in any other undischarged assumptions).

Our conclusion $\neg\forall x Px$ is a negated statement, so we prove it by assuming $\forall x Px$ and deriving a contradiction. Since $\exists x \neg Px$ gives us an assumption of $\neg P$, we have a contradiction if we can derive Pa . This follows from $\forall x Px$ by $\forall\text{Elim}$.

Existential 18

$$\frac{\frac{\frac{[\neg Pa]}{\exists x \neg Px} \exists \text{Intro} \quad [\neg \exists x \neg Px]}{\quad} \neg \text{Elim}}{\frac{Pa}{\forall x Px} \forall \text{Intro} \quad \neg \forall x Px} \neg \text{Elim}}{\exists x \neg Px} \neg \text{Elim}$$

Our conclusion $\exists x \neg Px$ is an existential statement, but it turns out we have no way of proving it directly. We have to assume $\neg \exists x \neg Px$ and show that it leads to a contradiction. Our premise $\neg \forall x Px$ is a negated statement, so we have the contradiction we need if we can derive $\forall x Px$.

This is a universal statement, so we can prove it if we can prove Pa without a appearing in any undischarged assumptions. Unfortunately, we have no way of proving Pa directly either: we have to assume $\neg Pa$ and show that this leads to a contradiction.

Because $\neg \exists x \neg Px$ is a negated statement, we have a contradiction if we can derive $\exists x \neg Px$. This follows from $\neg Pa$ by $\exists \text{Intro}$. Because the first application of $\neg \text{Elim}$ discharges $\neg Pa$ we are free to apply $\forall \text{Intro}$ and derive $\forall x Px$. This then gives us the contradiction which allows us to derive $\exists x \neg Px$.

Existential 19

$$\frac{\frac{[Pa] \quad \frac{\forall x \neg Px}{\neg Pa} \forall \text{Elim}}{\neg \exists x Px} \neg \text{Intro}}{\frac{[\exists x Px] \quad \neg \exists x Px}{\neg \exists x Px} \exists \text{Elim}}{\neg \exists x Px} \exists \text{Elim}} \quad \frac{[\exists x Px]}{\neg \exists x Px} \neg \text{Intro}$$

Our conclusion $\neg \exists x Px$ is a negated statement, so we assume $\exists x Px$ and try to derive a contradiction. $\exists x Px$ is an existential statement, so we can use it to discharge a proof of Pa (as long as a doesn't appear in any other undischarged assumptions when we apply $\exists \text{Elim}$).

Because we can derive $\neg Pa$ from our premise $\forall x \neg Px$ we have the contradiction we need, but applying $\neg \text{Intro}$ at this stage won't let us discharge $\exists x Px$. This means after applying $\exists \text{Elim}$ we assume $\exists x Px$ again and apply $\neg \text{Intro}$ a second time; this second application lets us derive the conclusion $\neg \exists x Px$ and discharge both assumptions of $\exists x Px$.

Existential 20

$$\frac{\frac{\frac{[Pb]}{\exists yPy} \exists \text{Intro} \quad \frac{\forall x(\exists yPy \rightarrow Qx)}{\exists yPy \rightarrow Qa} \forall \text{Elim}}{\rightarrow \text{Elim}}}{\frac{Qa}{Pb \rightarrow Qa} \rightarrow \text{Intro}} \exists \text{Intro} \quad \frac{\exists y(Py \rightarrow Qa)}{\forall x \exists y(Py \rightarrow Qx)} \forall \text{Intro}$$

This is a past paper question from 2011. The statement we want to prove is $\forall x \exists y (Py \rightarrow Qx)$, a universal statement. This means we can prove it from $\exists y (Py \rightarrow Qa)$, as long as a doesn't appear in any undischarged assumptions. This is an existential statement, so there are lots of statements we can prove it from. We'll try $Pb \rightarrow Qa$.

$Pb \rightarrow Qa$ is an implication, so we prove it by assuming Pb and trying to derive Qa . We can get to Qa with the help of our premise, $\forall x(\exists yPy \rightarrow Qx)$. From this we can derive $\exists yPy \rightarrow Qa$, which is an implication with Qa as its consequent. All we need to do is prove $\exists yPy$; fortunately this follows from our assumption of Pb .

The proof would also have worked if we had assumed Pa at the very top of the proof, and then derived $\exists y(Py \rightarrow Qa)$ from $Pa \rightarrow Qa$. Our \forall Intro step would still have been allowed because the assumption of Pa would have been discharged before applying \forall Intro. This alternate proof is shown below:

$$\frac{\frac{\frac{[Pa]}{\exists yPy} \exists \text{Intro} \quad \frac{\forall x(\exists yPy \rightarrow Qx)}{\exists yPy \rightarrow Qa} \forall \text{Elim}}{\rightarrow \text{Elim}}}{\frac{Qa}{Pa \rightarrow Qa} \rightarrow \text{Intro}} \exists \text{Intro} \quad \frac{\exists y(Py \rightarrow Qa)}{\forall x \exists y(Py \rightarrow Qx)} \forall \text{Intro}$$

Existential 21

$$\frac{\frac{\frac{[Pab]}{\exists y Pay} \exists \text{Intro} \quad [\neg \exists y Pay]}{Qab} \neg \text{Elim}}{Pab \rightarrow Qab} \rightarrow \text{Intro} \quad \frac{\forall x \neg \forall y (Pxy \rightarrow Qxy)}{\neg \forall y (Pay \rightarrow Qay)} \forall \text{Elim}}{\frac{\forall y (Pay \rightarrow Qay)}{\exists y Pay} \forall \text{Intro} \quad \frac{\neg \forall y (Pay \rightarrow Qay)}{\exists y Pay} \neg \text{Elim}}{\forall x \exists y Pxy} \forall \text{Intro}$$

This is a past paper question from 2013. Our conclusion $\forall x \exists y Pxy$ is a universal statement, so we can prove it from $\exists y Pay$ as long as a doesn't appear in any undischarged assumptions. This is an existential statement, but we have no way of proving it directly. Instead we have to assume $\neg \exists y Pay$ and show that it leads to a contradiction. Since our premise $\forall x \neg \forall y (Pxy \rightarrow Qxy)$ gives us the negated statement $\neg \forall y (Pay \rightarrow Qay)$, we have the contradiction we need if we can provide a proof of $\forall y (Pay \rightarrow Qay)$.

$\forall y (Pay \rightarrow Qay)$ is a universal statement, so we can prove it from $Pab \rightarrow Qab$, as long as b doesn't appear in any undischarged assumptions. Remember that the conditions for $\forall \text{Intro}$ prevent us from using a twice. $Pab \rightarrow Qab$ is an implication, so we can assume Pab and need to prove Qab . We can't obtain Qab from anything directly, but from our assumption of Pab we can obtain $\exists y Pay$, which contradicts our assumption of $\neg \exists y Pay$. This lets us obtain Qab by $\neg \text{Elim}$.

Existential 22

$$\frac{\frac{\forall x (Pxx \vee \forall y Qxy)}{Paa \vee \forall y Qay} \forall \text{Elim} \quad \frac{\frac{[Paa]}{\exists y Pay} \exists \text{Intro} \quad \frac{[\forall y Qay]}{Qaa} \forall \text{Elim}}{\exists y Pay \vee Qaa} \forall \text{Intro}}{\exists y Pay \vee Qaa} \forall \text{Elim}}{\forall x (\exists y Pxy \vee Qxx)} \forall \text{Intro}$$

This is a past paper question from 2009. Our conclusion $\forall x (\exists y Pxy \vee Qxx)$ is a universal statement, so we can prove it from $\exists y Pay \vee Qaa$ as long as a doesn't appear in any undischarged assumptions. Our premise $\forall x (Pxx \vee \forall y Qxy)$ gives us $Paa \vee \forall y Qay$, a disjunction splitting the proof into a case where Paa is true and a case where $\forall y Qay$ is true. In the left-hand case we can obtain $\exists y Pay$ by $\exists \text{Intro}$, and in the right-hand case we can obtain Qaa by $\forall \text{Elim}$. This means that in both cases we have proofs of $\exists y Pay \vee Qaa$.

Existential 23

$$\frac{\frac{\frac{[Paa \wedge \forall yQay]}{Paa} \wedge\text{Elim} \quad \frac{[Paa \wedge \forall yQay]}{\forall yQay} \wedge\text{Elim}}{\exists yPay} \exists\text{Intro} \quad \frac{\forall yQay}{Qaa} \forall\text{Elim}}{\exists yPay \wedge Qaa} \wedge\text{Intro}}{\exists x(Pxx \wedge \forall yQxy) \quad \frac{\exists x(\exists yPay \wedge Qaa)}{\exists x(\exists yPay \wedge Qaa)} \exists\text{Intro}} \exists\text{Elim}$$

This is a past paper question from 2009. Our premise $\exists x(Pxx \wedge \forall yQxy)$ is an existential statement, so we can apply $\exists\text{Elim}$ at the end of the proof to discharge assumptions of $Paa \wedge \forall yQay$, provided a doesn't appear in any other undischarged assumptions. Our conclusion $\exists x(\exists yPay \wedge Qaa)$ is an existential statement; since our premise is going to discharge assumptions involving a , it makes sense to try to derive $\exists x(\exists yPay \wedge Qaa)$ from $\exists yPay \wedge Qaa$.

This is a conjunction, so we need to provide a proof of $\exists yPay$ and a proof of Qaa . Both of these can be derived from our assumption of $Paa \wedge \forall yQay$. From Paa we can derive $\exists yPay$ by $\exists\text{Intro}$, and from $\forall yQay$ we can derive Qaa by $\forall\text{Elim}$.

Existential 24

$$\begin{array}{c}
 \frac{[\neg\forall xRxx]}{\forall x\exists yRxy \vee \neg\forall xRxx} \vee\text{Intro} \quad \frac{[\neg(\forall x\exists yRxy \vee \neg\forall xRxx)]}{\neg\forall xRxx} \neg\text{Elim} \\
 \frac{\frac{\frac{\frac{\forall xRxx}{\forall xRxx} \vee\text{Elim}}{Raa} \exists\text{Intro}}{\exists yRay} \vee\text{Intro}}{\forall x\exists yRxy} \vee\text{Intro}}{\forall x\exists yRxy \vee \neg\forall xRxx} \vee\text{Intro} \quad \frac{[\neg(\forall x\exists yRxy \vee \neg\forall xRxx)]}{\neg\forall xRxx} \neg\text{Elim} \\
 \frac{\forall x\exists yRxy \vee \neg\forall xRxx}{\forall x\exists yRxy \vee \neg\forall xRxx} \neg\text{Elim}
 \end{array}$$

This is a past paper question from 2013. Our conclusion $\forall x\exists yRxy \vee \neg\forall xRxx$ is a disjunction which we are asked to prove from no premises. This makes it pretty likely that we'll need to carry out an indirect proof, assuming $\neg(\forall x\exists yRxy \vee \neg\forall xRxx)$ and showing that it leads to a contradiction.

We follow the usual strategy for indirectly proving disjunctions. First we assume one disjunct, $\neg\forall xRxx$, and derive the conclusion $\forall x\exists yRxy \vee \neg\forall xRxx$ from it. Using our assumption of $\neg(\forall x\exists yRxy \vee \neg\forall xRxx)$ we apply $\neg\text{Elim}$ and derive $\forall xRxx$. With this proof of $\forall xRxx$ we want to derive the other disjunct, $\forall x\exists yRxy$.

This is a universal statement, so we can derive it from $\exists yRay$ (as long as a doesn't appear in any undischarged assumptions). This in turn can be derived from Raa , which can be derived by $\forall\text{Elim}$ from $\forall xRxx$.

From $\forall x\exists yRxy$ we derive $\forall x\exists yRxy \vee \neg\forall xRxx$ a second time and assume $\neg(\forall x\exists yRxy \vee \neg\forall xRxx)$ a second time. Finally we apply $\neg\text{Elim}$, discharging both assumptions of $\neg(\forall x\exists yRxy \vee \neg\forall xRxx)$ and deriving $\forall x\exists yRxy \vee \neg\forall xRxx$.

An alternate proof, starting from $\forall x\exists yRxy$, is shown below:

$$\begin{array}{c}
 \frac{\frac{[\forall x\exists yRxy]}{\forall x\exists yRxy \vee \neg\forall xRxx} \vee\text{Intro} \quad \frac{[\neg(\forall x\exists yRxy \vee \neg\forall xRxx)]}{\neg\forall x\exists yRxy} \neg\text{Intro}}{\neg\forall x\exists yRxy} \neg\text{Intro} \quad \frac{\frac{[\forall xRxx]}{\forall xRxx} \vee\text{Elim}}{Raa} \exists\text{Intro}}{\exists yRay} \vee\text{Intro}}{\forall x\exists yRxy} \vee\text{Intro} \\
 \frac{\frac{\neg\forall xRxx}{\forall x\exists yRxy \vee \neg\forall xRxx} \vee\text{Intro}}{\forall x\exists yRxy \vee \neg\forall xRxx} \vee\text{Intro} \quad \frac{[\neg(\forall x\exists yRxy \vee \neg\forall xRxx)]}{\neg\forall xRxx} \neg\text{Elim} \\
 \frac{\forall x\exists yRxy \vee \neg\forall xRxx}{\forall x\exists yRxy \vee \neg\forall xRxx} \neg\text{Elim}
 \end{array}$$

Existential 25

$$\frac{\frac{\frac{[\forall y Rya]}{\forall y Rya} \forall\text{Elim}}{\exists y Rcy} \exists\text{Intro}}{\forall x \exists y Rxy} \forall\text{Intro} \quad \frac{\frac{[Rbb]}{\exists x Rxx} \exists\text{Intro} \quad \frac{\forall x \exists y Rxy \rightarrow \neg \exists x Rxx}{\neg \exists x Rxx} \rightarrow\text{Elim}}{\neg \exists x Rxx} \neg\text{Intro}}{\frac{\frac{\exists x \forall y Ryx}{\forall x \neg Rxx} \exists\text{Elim} \quad \frac{\frac{\neg Rbb}{\forall x \neg Rxx} \forall\text{Intro}}{\forall x \neg Rxx} \exists\text{Elim}}{\forall x \neg Rxx} \exists\text{Elim}}$$

This is adapted from a past paper question from 2010. Our premise $\exists x \forall y Ryx$ is an existential statement, letting us discharge assumptions of $\forall y Rya$, as long as a doesn't appear in any other undischarged assumptions by the end of the proof.

We want to prove $\forall x \neg Rxx$, which we can derive from $\neg Rbb$ (as long as b doesn't appear in any undischarged assumptions when we apply $\forall\text{Intro}$). This is a negated statement, so we prove it by assuming Rbb and showing that it leads to a contradiction.

We don't have any negated statements readily available, but our other premise $\forall x \exists y Rxy \rightarrow \neg \exists x Rxx$ is a conditional with a negated consequent. This means that if we can prove $\forall x \exists y Rxy$ we will be able to derive $\neg \exists x Rxx$ by $\rightarrow\text{Elim}$.

$\forall x \exists y Rxy$ follows from our assumption of $\forall y Rya$, so we have a proof of $\neg \exists x Rxx$. Since $\exists x Rxx$ follows from Rbb , we have the contradiction we need to discharge Rbb and apply $\neg\text{Intro}$. This then lets us derive our conclusion $\forall x \neg Rxx$; finally an $\exists\text{Elim}$ step discharges our assumption of $\forall y Rya$.

Existential 26

$$\frac{\frac{\frac{[\neg\forall xPx]}{\forall x\exists zRzx \vee \neg\forall xPx} \vee I \quad [\neg(\forall x\exists zRzx \vee \neg\forall xPx)]}{\frac{\forall xPx}{Pa} \vee E} \neg E \quad \frac{\frac{\forall x(Px \rightarrow \exists yRyx)}{Pa \rightarrow \exists yRya} \vee E \quad \frac{[Rba]}{\exists zRza} \exists I}{\exists yRya} \rightarrow E \quad \frac{[Rba]}{\exists zRza} \exists E}{\frac{\frac{\exists zRza}{\forall x\exists zRzx} \vee I}{\forall x\exists zRzx \vee \neg\forall xPx} \vee I \quad [\neg(\forall x\exists zRzx \vee \neg\forall xPx)]}{\forall x\exists zRzx \vee \neg\forall xPx} \neg E}$$

This is adapted from a past paper question from 2013. This is another indirect proof of a disjunction, so we follow the usual strategy of starting by assuming $\neg\forall xPx$, one of the disjuncts, deriving the conclusion by \vee Intro and then assuming the negation of the conclusion to create a contradiction. With this contradiction we apply \neg Elim to derive $\forall xPx$.

The legwork of the proof lies in using our premise $\forall x(Px \rightarrow \exists yRyx)$ to derive $\forall x\exists zRzx$ from $\forall xPx$. $\forall x\exists zRzx$ is a universal statement, which we can derive from $\exists zRza$ (as long as a doesn't appear in any undischarged assumptions when we apply \vee Intro). From $\forall xPx$ we can derive Pa and from $\forall x(Px \rightarrow \exists yRyx)$ we can derive $Pa \rightarrow \exists yRya$, meaning we can apply \rightarrow Elim to derive $\exists yRya$. This is almost what we need, but we need to apply \exists Intro and \exists Elim to convert it to $\exists zRza$.

With $\forall x\exists zRzx$ we can apply \vee Intro again to derive the conclusion. We assume the negation of the conclusion a second time and apply \neg Elim to discharge both assumptions of $\neg(\forall x\exists zRzx \vee \neg\forall xPx)$ and derive $\forall x\exists zRzx \vee \neg\forall xPx$.

Existential 27

$$\begin{array}{c}
 \frac{\frac{\frac{\forall x \forall y \forall z (Rxy \vee Rzy \vee Rzx)}{\forall y \forall z (Ray \vee Rzy \vee Rza)} \forall E}{\forall z (Rab \vee Rzb \vee Rza)} \forall E}{Rab \vee Rcb \vee Rca} \forall E \quad \frac{[Rab] \vee Rcb]^2 \quad \frac{[Rab]^1 \quad [\neg Rab]^3}{Rca \vee Rcb} \neg E \quad \frac{[Rcb]^1}{Rca \vee Rcb} \forall I}{Rca \vee Rcb} \forall E^1 \quad \frac{[Rca]^2}{Rca \vee Rcb} \forall I}{Rca \vee Rcb} \forall E^2 \\
 \frac{\frac{\frac{Rca \vee Rcb}{\forall z (Rza \vee Rzb)} \forall I}{\exists y \forall z (Rza \vee Rzy)} \exists I}{\exists x \exists y \forall z (Rzx \vee Rzy)} \exists I \\
 \vdots \\
 \vdots \\
 \frac{\frac{\frac{\frac{\frac{\exists x \exists y \forall z (Rzx \vee Rzy)}{\exists x \exists y \forall z (Rzx \vee Rzy)} \exists I}{\exists y \forall z (Rza \vee Rzy)} \exists I}{\forall z (Rza \vee Rzb)} \forall I}{\frac{Rab}{Rab \vee Rbc} \forall I}{Rab \vee Rbc} \forall I}{\exists x \exists y \forall z (Rzx \vee Rzy)} \exists I \quad \frac{[\neg \exists x \exists y \forall z (Rzx \vee Rzy)]^4}{\exists x \exists y \forall z (Rzx \vee Rzy)} \neg E^4}{\exists x \exists y \forall z (Rzx \vee Rzy)} \neg E^3
 \end{array}$$

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Unfortunately, the above proof is too wide to fit on a single page.

The first thing we should notice is what's similar between our premise and our conclusion. The premise is $\forall x \forall y \forall z (Rxy \vee Rzy \vee Rzx)$, so from it we can derive $Rab \vee Rcb \vee Rca$. The conclusion is $\exists x \exists y \forall z (Rzx \vee Rzy)$, so we can derive it from $Rca \vee Rcb$, provided c doesn't appear in any undischarged assumptions by the time we derive $Rca \vee Rcb$.

Deriving $Rca \vee Rcb$ from $Rab \vee Rcb \vee Rca$ is possible if we can provide a proof of $\neg Rab$. We don't have a proof of $\neg Rab$, but instead we can simply assume it. Deriving $Rca \vee Rcb$ then is a straightforward (if a little fiddly) case of applying \vee Elim and \vee Intro, applying \neg Elim in the Rab case. Notice that, according to the bracketing conventions, $Rab \vee Rcb \vee Rca$ is an abbreviation of $((Rab \vee Rcb) \vee Rca)$, which is why we need to use \vee Elim once to discharge assumptions of Rab and Rcb and a second time to discharge assumptions of $Rab \vee Rcb$ and Rca .

After deriving $Rca \vee Rcb$, we still have $\neg Rab$ undischarged. Hence we can apply \forall Intro to obtain $\forall z(Rza \vee Rzb)$, but we wouldn't be able to derive (for example) $\forall x \forall y \forall z (Rzx \vee Rzy)$.

Once we have $\exists x \exists y \forall z (Rzx \vee Rzy)$ (which is where the proof above splits in two), we need to discharge our assumption of $\neg Rab$, so we assume $\neg \exists x \exists y \forall z (Rzx \vee Rzy)$ and discharge our assumption of $\neg Rab$ by \neg Elim. We now have a proof of Rab with no undischarged assumptions involving of a , b or c ; hence we have no problem deriving $\exists x \exists y \forall z (Rzx \vee Rzy)$ a second time. Then we assume $\neg \exists x \exists y \forall z (Rzx \vee Rzy)$ once more, discharge both assumptions of $\neg \exists x \exists y \forall z (Rzx \vee Rzy)$ by \neg Elim, and derive $\exists x \exists y \forall z (Rzx \vee Rzy)$.

Bonus challenge

The following proof has the two features specified in the challenge:

$$\frac{\exists x P \quad [P]}{P} \exists\text{Elim}$$

This requires us to apply \neg Elim in a bizarre way, but it is indeed allowed. Recall the formulation of the \exists Elim rule:

$$\frac{\begin{array}{c} [\phi[t/v]] \\ \vdots \\ \exists v \phi \quad \psi \end{array}}{\psi} \exists\text{Elim}$$

The \exists Elim rule lets us discharge all assumptions of $\phi[t/v]$, where $\phi[t/v]$ is the result of replacing all occurrences of v (x in this case) in ϕ (P in this case) with the constant t . But here there are no occurrences of x in P , P is what we discharge. ψ happens to be P as well in this case, so P is what we conclude.

Furthermore, all additional conditions for \exists Elim are satisfied: there are no constants in the proof at all, so none appear in ψ or ϕ or in any undischarged assumptions in the proof of ψ .

This is an absurd proof, but it highlights an unusual way in which the quantifier rules can be applied. Similar proofs exist for \exists Intro and the two rules for the universal quantifier:

$$\frac{P}{\exists x P} \exists\text{Intro} \quad \frac{\forall x P}{P} \forall\text{Elim} \quad \frac{P}{\forall x P} \forall\text{Intro}$$

5.9 Identity

Identity 1

$$\frac{\frac{[a = a]^=}{\exists y a = y} \exists\text{Intro}}{\exists x \exists y x = y} \exists\text{Intro}$$

We aren't given any premises, but we can apply =Intro to make and immediately discharge the assumption $a = a$. From this, we can derive $\exists x \exists y x = y$ by applying \exists Intro twice.

Identity 2

$$\frac{\frac{[b = b]^=}{b = b \wedge b = c} \wedge\text{Intro} \quad \frac{a = b \quad [a = c]}{b = c} =\text{Elim}}{\neg a = c} \neg\text{Intro}$$

Our conclusion $\neg a = c$ is negated, so we can derive it by assuming $a = c$ and showing it leads to a contradiction. Since one of our premises $\neg(b = b \wedge b = c)$ is negated, we have the contradiction we need if we can provide a proof of $b = b \wedge b = c$.

$b = b$ is easy to prove: we can assume it and immediately discharge it by =Intro. To obtain $b = c$, we apply =Elim using our premise $a = b$ and our assumption of $a = c$.

Identity 3

$$\frac{\frac{\frac{[c = a] \quad [a = b]}{c = b} =\text{Elim}}{c = a \rightarrow c = b} \rightarrow\text{Intro} \quad \frac{[\forall x(x = a \rightarrow x = b)]}{a = a \rightarrow a = b} \forall\text{Elim}}{\forall x(x = a \rightarrow x = b)} \forall\text{Intro} \quad \frac{[a = a] = \quad a = a \rightarrow a = b}{a = b} \rightarrow\text{Elim}}{a = b \leftrightarrow \forall x(x = a \rightarrow x = b)} \leftrightarrow\text{Intro}$$

Our conclusion is a biconditional, so we need to provide two proofs: one proof of $\forall x(x = a \rightarrow x = b)$ from $a = b$ and one of $a = b$ from $\forall x(x = a \rightarrow x = b)$.

On the left-hand side, we want to derive the universal statement $\forall x(x = a \rightarrow x = b)$. We can derive this from $c = a \rightarrow c = b$, provided c doesn't appear in any undischarged assumptions in our proof of $c = a \rightarrow c = b$. Because of the restrictions on $\forall\text{Intro}$ we can't use a or b instead of c : a and b both appear in $\forall x(x = a \rightarrow x = b)$, and our assumption $a = b$ won't be discharged until the end of the proof. $c = a \rightarrow c = b$ is an implication, so we assume $c = a$ and apply $=\text{Elim}$ using our assumption of $a = b$ to derive $c = b$.

On the right-hand side, we want to derive $a = b$. We can't derive $a = b$ using the introduction rule for $=$, but our assumption of $\forall x(x = a \rightarrow x = b)$ helps us: we can apply $\forall\text{Elim}$ to obtain $a = a \rightarrow a = b$ and then derive $a = a$ by $=\text{Intro}$, giving us $a = b$ by $\rightarrow\text{Elim}$.

Notice the symmetry in the proof we obtain: the left-hand side uses $=\text{Elim}$, $\rightarrow\text{Intro}$ and $\forall\text{Intro}$ while the right-hand side uses $=\text{Intro}$, $\rightarrow\text{Elim}$ and $\forall\text{Elim}$.

Identity 4

$$\frac{\frac{[\forall ya = y]^1}{a = b} \forall\text{Elim} \quad \frac{[\forall ya = y]^1}{a = c} \forall\text{Elim}}{b = c} =\text{Elim} \quad \frac{b = c}{\forall yb = y} \forall\text{Intro} \quad \frac{[\forall x\forall yx = y]^2}{\forall x\forall yx = y} \forall\text{Intro}}{\exists x\forall yx = y} \exists\text{Elim}^1 \quad \frac{[\forall x\forall yx = y]^2}{\forall ya = y} \forall\text{Elim} \quad \frac{\forall ya = y}{\exists x\forall yx = y} \exists\text{Intro}}{\exists x\forall yx = y \leftrightarrow \forall x\forall yx = y} \leftrightarrow\text{Intro}^2$$

This is a past paper question from 2009. Our conclusion is a biconditional, so we need to provide a proof of $\forall x\forall yx = y$ from $\exists x\forall yx = y$ and a proof of $\exists x\forall yx = y$ from $\forall x\forall yx = y$.

The right-hand side is easy: we use $\forall\text{Elim}$ to replace x with any constant (a is used in the proof above) and then we use $\exists\text{Intro}$ to replace that constant with x again.

On the left-hand side, we have an existential assumption $\exists x\forall yx = y$ which lets us discharge an assumption of $\forall ya = y$. We want to prove $\forall x\forall yx = y$, which we can derive from $b = c$. We can't derive it from anything involving a because our assumption of $\forall ya = y$ won't be discharged when we apply $\forall\text{Intro}$. We need to use our assumption of $\forall ya = y$ twice, deriving $a = b$ and $a = c$. Then we can apply $=\text{Elim}$ to derive $b = c$.

Identity 5

$$\frac{\frac{Pa \quad [a = b]}{Pb} =\text{Elim} \quad \neg Pb}{\neg a = b} \neg\text{Intro}$$

Our conclusion $\neg a = b$ is a negation, so we derive it by assuming $a = b$ and trying to derive a contradiction. One of our premises $\neg Pb$ is a negation, so we have a contradiction if we can derive Pb .

To obtain Pb we need to make use of our assumption of $a = b$ and apply $=\text{Elim}$, replacing the a in Pa (our other premise) with b .

We could also have carried out the proof in a different way, applying $=\text{Elim}$ to obtain $\neg Pa$ which contradicts Pa :

$$\frac{Pa \quad \frac{\neg Pb \quad [a = b]}{\neg Pa} =\text{Elim}}{\neg a = b} \neg\text{Intro}$$

Identity 6

$$\frac{\frac{Pb \wedge Qb}{Qb} \wedge\text{Elim} \quad \frac{\frac{Pb \wedge Qb}{Pb} \wedge\text{Elim} \quad \frac{\forall x(Px \rightarrow x = a)}{Pb \rightarrow b = a} \forall\text{Elim}}{b = a} \rightarrow\text{Elim}}{Qa} =\text{Elim}$$

Our conclusion is Qa . This doesn't have any connectives or quantifiers in it, so we know the last line of our proof won't be an introduction rule. Qa doesn't appear explicitly in any of our premises, and we can't derive it using $\neg\text{Elim}$ (assuming $\neg Qa$ is no help).

However, we do have the premise $Pb \wedge Qb$, which gives us Qb ; we can derive Qa by $=\text{Elim}$ if we can prove $b = a$. We can show $b = a$ by $\rightarrow\text{Elim}$, since our first premise gives us Pb and our other premise gives us $\forall x(Px \rightarrow x = a)$ gives us $Pb \rightarrow b = a$.

Identity 7

$$\frac{\frac{\frac{[\forall y(Py \leftrightarrow a = y)]}{Pb \leftrightarrow a = b} \forall\text{Elim}}{a = b} \leftrightarrow\text{Elim}}{Pb \rightarrow a = b} \rightarrow\text{Intro}
 }{\forall y(Py \rightarrow a = y)} \forall\text{Intro}
 }{\frac{\exists x\forall y(Py \leftrightarrow x = y) \quad \exists x\forall y(Py \rightarrow x = y)}{\exists x\forall y(Py \rightarrow x = y)} \exists\text{Elim}} \exists\text{Intro}$$

This is a past paper question from 2011. It turns out we don't need to use any of the rules for identity, but we do need all four quantifier rules.

Because we have an existential premise $\exists x\forall y(Py \leftrightarrow x = y)$, we should look at this first and apply $\exists\text{Elim}$ at the end of the proof to discharge $\forall y(Py \leftrightarrow a = y)$ (making sure that a doesn't appear in any other undischarged assumptions by the time we apply $\exists\text{Elim}$).

Our conclusion $\exists x\forall y(Py \rightarrow x = y)$ is also an existential statement, and it's reasonable to suspect that we'll derive it from $\forall y(Py \rightarrow a = y)$ (that both the premise and the conclusion refer to the same object). This is a universal statement, so we'll derive it from $Pb \rightarrow a = b$. This is an implication, so we assume Pb and try to derive $a = b$. We do this by $\leftrightarrow\text{Elim}$ using $Pb \leftrightarrow a = b$, which can be derived from our assumption of $\forall y(Py \leftrightarrow a = y)$.

Identity 8

$$\frac{\frac{\frac{\forall x(x = a \vee x = b)}{c = a \vee c = b} \forall E \quad \frac{\frac{[Pc]^3 \ [c = a]^1}{Pa} =E \quad \frac{[\neg Pa]^2}{Pb} \neg E \quad \frac{[Pc]^3 \ [c = b]^1}{Pb} =E}{Pb} \vee E^1}{\frac{\exists x Px}{\neg Pa \rightarrow Pb} \rightarrow I^2} \exists E^3$$

One of our premises $\exists x Px$ is an existential statement, so we should think about that first. If we apply \exists Elim at the end of the proof we can discharge assumptions of Pc , as long as c doesn't appear in any undischarged assumptions. We can't use a or b because they both appear in our conclusion $\neg Pa \rightarrow Pb$.

$\neg Pa \rightarrow Pb$ is an implication, so we derive it by assuming $\neg Pa$ and deriving Pb from it. Our premise $\forall x(x = a \vee x = b)$ is a universal statement. We can derive lots of disjunctions from it, but not all of them will be useful. $a = a \vee a = b$, for example, is something we could derive without any premises if we wanted to: when we can use $=$ Intro to assume and discharge $a = a$ and then use \vee Intro to derive $a = a \vee a = b$.

$c = a \vee c = b$ is useful: it splits the proof into a case where $c = a$ is true and a case where $c = b$ is true. On the right-hand side, our assumptions of Pc and $c = b$ let us derive Pb by $=$ Elim. On the left-hand side, we can use $=$ Elim to derive Pa , which contradicts our other assumption of $\neg Pa$ and lets us derive Pb by \neg Elim.

Identity 10

$$\frac{\frac{[Pa \wedge Qa]}{Qa} \wedge\text{Elim} \quad \frac{\frac{[Pa \wedge Qa]}{Pa} \wedge\text{Elim} \quad [a = a]^= \quad \frac{\forall x \forall y (Px \wedge x = y \rightarrow \neg Qy)}{\forall y (Pa \wedge a = y \rightarrow \neg Qy)} \forall\text{Elim}}{Pa \wedge a = a} \wedge\text{Intro} \quad \frac{\forall y (Pa \wedge a = y \rightarrow \neg Qy)}{Pa \wedge a = a \rightarrow \neg Qa} \forall\text{Elim}}{\neg Qa} \rightarrow\text{Elim}
 }{\frac{\neg(Pa \wedge Qa)}{\forall z \neg (Pz \wedge Qz)} \forall\text{Intro}} \neg\text{Intro}$$

This is a past paper question from 2012. Our conclusion is a universal statement, so we can derive it from $\neg(Pa \wedge Qa)$ (as long as a doesn't appear in any undischarged assumptions by the end of the proof). This is a negated statement, so we assume $Pa \wedge Qa$ and show that it leads to a contradiction.

We don't have any negated statements immediately available which can give us the contradiction we need, but our premise $\forall y(Pa \wedge a = y \rightarrow \neg Qy)$ can give us one. If we apply $\forall\text{Elim}$ twice, we can obtain $Pa \wedge a = a \rightarrow \neg Qa$, which is an abbreviation of $(Pa \wedge a = a) \rightarrow \neg Qa$. This means that if we can prove $Pa \wedge a = a$ we can prove $\neg Qa$ by $\rightarrow\text{Elim}$.

Pa comes from our assumption of $Pa \wedge Qa$ and $a = a$ can be assumed and discharged by $=\text{Intro}$, so a proof of $Pa \wedge a = a$ is easy to provide. With a proof of $\neg Qa$ and a proof of Qa (which also follows from our assumption of $Pa \wedge Qa$) we have the contradiction we need.

Identity 11

$$\frac{[a = a] = \frac{\frac{\frac{\forall x \forall y (Rxy \leftrightarrow x = y)}{\forall y (Ray \leftrightarrow a = y)} \forall\text{Elim}}{Raa \leftrightarrow a = a} \forall\text{Elim}}{\frac{Raa}{\forall x Rxx} \forall\text{Intro}} \leftrightarrow\text{Elim}}{\forall x Rxx}$$

Our conclusion $\forall x Rxx$ is a universal statement, so we can derive it from Raa (as long as a appears in no undischarged assumptions). By applying $\forall\text{Elim}$ twice on our premise we can derive $Raa \leftrightarrow a = a$; this allows us to apply $\leftrightarrow\text{Elim}$ and derive Raa if we can provide a proof of $a = a$. $=\text{Intro}$ gives us the proof of $a = a$ we need by allowing us to make and immediately discharge an assumption of $a = a$.

Identity 12

$$\frac{\frac{\frac{Rab \quad [a = b]}{Raa} =\text{Elim} \quad \frac{\frac{\forall x \neg Rxx}{\neg Raa} \forall\text{Elim}}{\neg a = b} \neg\text{Intro}}{\exists y \neg a = y} \exists\text{Intro}}{\exists x \exists y \neg x = y} \exists\text{Intro}}$$

This is a past paper question from 2012. The conclusion $\exists x \exists y \neg x = y$ has two existential quantifiers, so it's likely we will derive it by applying $\exists\text{Intro}$, but we also need to determine which statement we should try to derive it from. The constants a and b appear in the premise Rab , so $\neg a = b$ would be a good sentence to try and prove.

This is a negation, so we prove it by assuming $a = b$ and showing that it leads to a contradiction. There are actually lots of ways we can derive a contradiction from this assumption and our premises Rab and $\forall x \neg Rxx$. In the proof above, we use $=\text{Elim}$ and $a = b$ to replace the b in Rab with a , giving Raa . Because we can derive $\neg Raa$ from $\forall x \neg Rxx$, we have the contradiction we need.

Identity 13

$$\begin{array}{c}
 \frac{\frac{[Rbc]^1 \quad \frac{[\forall ya = y]^3}{a = b} \forall\text{Elim}}{Rac} =\text{Elim} \quad \frac{[\forall ya = y]^3}{a = c} \forall\text{Elim}}{Raa} =\text{Elim} \quad \frac{[\forall ya = y]^3}{a = b_1} \forall\text{Elim} \quad \frac{[\forall ya = y]^3}{a = c_1} \forall\text{Elim}}{Rb_1a} =\text{Elim} \\
 \frac{\frac{\frac{[\exists yRby]^2}{\exists x\exists yRxy} \quad \frac{\frac{Rb_1c_1}{\forall yRb_1y} \forall\text{Intro} \quad \frac{\forall x\forall yRxy}{\forall x\forall yRxy} \forall\text{Intro}}{\exists\text{Elim}^1} \quad \frac{\exists x\exists yRxy}{\forall x\forall yRxy} \exists\text{Elim}^2}}{\exists x\forall yx = y} \exists\text{Elim}^3}{\forall x\forall yRxy}
 \end{array}$$

We have two existential premises, so we should think about those first. From $\exists x\forall yx = y$ we can apply $\exists\text{Elim}$ to discharge an assumption of $\forall ya = y$, and from $\exists x\exists yRxy$ we can apply $\exists\text{Elim}$ twice to discharge an assumption of Rbc .

Our conclusion is $\forall x\forall yRxy$, a universal statement, so we derive it from Rb_1c_1 . We can't derive it from any statements involving a , b or c because they appear in our assumptions of Rbc and $\forall ya = y$, which won't be discharged until the very end of the proof.

Now all we need to do is move from our assumptions of Rbc and $\forall ya = y$ to Rb_1c_1 . This is easy to do using $=\text{Elim}$, because our assumption of $\forall ya = y$ tells us that everything is identical to a . First we replace the b and c with Rbc with as , and then we replace those as with b_1 and c_1 .

Identity 14

$$\frac{\frac{\frac{Paa}{\forall x Pxx} \forall \text{Elim} \quad \frac{[\neg Pab]^2 \quad [a = b]^1}{\neg Paa} =\text{Elim}}{\neg a = b} \neg \text{Intro}^1 \quad \frac{\frac{\frac{\neg a = b}{\neg Pab \rightarrow \neg a = b} \rightarrow \text{Intro}^2}{\forall y(\neg Pay \rightarrow \neg a = y)} \forall \text{Intro}}{\forall x \forall y(\neg Pxy \rightarrow \neg x = y)} \forall \text{Intro}}{\forall x Pxx \leftrightarrow \forall x \forall y(\neg Pxy \rightarrow \neg x = y)} \leftrightarrow \text{Intro}^4}{\frac{\frac{[\forall x \forall y(\neg Pxy \rightarrow \neg x = y)]^4}{\forall y(\neg Pay \rightarrow \neg a = y)} \forall \text{Elim}}{\neg Paa \rightarrow \neg a = a} \forall \text{Elim} \quad \frac{[\neg Paa]^3}{\neg a = a} \rightarrow \text{Elim}}{\frac{Paa}{\forall x Pxx} \forall \text{Intro}} \neg \text{Elim}^3} [a = a] =$$

This is a past paper question from 2009. Our conclusion is a biconditional, so we need to provide a proof of $\forall x \forall y(\neg Pxy \rightarrow \neg x = y)$ from $\forall x Pxx$ and a proof of $\forall x Pxx \forall x \forall y(\neg Pxy \rightarrow \neg x = y)$.

On the left-hand side, we can derive $\forall x Pxx \forall x \forall y(\neg Pxy \rightarrow \neg x = y)$ by deriving $\neg Pab \rightarrow \neg a = b$ and applying $\forall \text{Intro}$ twice, as long as neither a or b appear in any undischarged assumptions when we apply $\forall \text{Intro}$. $\neg Pab \rightarrow \neg a = b$ is an implication, so we assume $\neg Pab$ and try to derive $\neg a = b$. Because $\neg a = b$ is a negated statement, we assume $a = b$ and try to derive a contradiction.

There are many ways we can derive a contradiction from $\forall x Pxx$, $\neg Pab$ and $a = b$. In the proof above, we use $a = b$ to replace the b in $\neg Pab$ with a , giving us $\neg Paa$. This contradicts Paa , which can be derived from $\forall x Pxx$.

On the right-hand side, we need to prove the universal statement $\forall x Pxx$, which can be derived from Paa (as long as a doesn't appear in any undischarged assumptions when we apply $\forall \text{Intro}$). Sadly we don't have any way of deriving Paa directly; instead we prove it by deriving a contradiction from assumptions of $\neg Paa$.

From our assumption of $\forall x Pxx \forall x \forall y(\neg Pxy \rightarrow \neg x = y)$ we can derive $\neg Paa \rightarrow \neg a = a$, so we can derive $\neg a = a$ by $\rightarrow \text{Elim}$. This gives us the contradiction we need, because $=\text{Intro}$ lets us make and immediately discharge an assumption of $a = a$.

Notice that this proof involves a particular symmetry: on the left-hand side we apply $\forall \text{Intro}$ (twice), $\rightarrow \text{Intro}$, $\neg \text{Intro}$, $=\text{Elim}$ and $\forall \text{Elim}$; on the right-hand side we apply $\forall \text{Elim}$ (twice), $\rightarrow \text{Elim}$, $\neg \text{Elim}$, $=\text{Intro}$ and $\forall \text{Intro}$. Each time an introduction rule appears on one side, the corresponding elimination rule appears on the other side.

5.10 Additional challenges

Admissible rules 1

The rule $\star 1$ is admissible. We can show that any proof making use of $\star 1$ can be rewritten using \wedge Elim.

Suppose we have a proof involving one or more applications of $\star 1$. Each application of $\star 1$ corresponds to a subproof of the following form:

$$\frac{\begin{array}{c} [\phi] \\ \vdots \\ \phi \wedge \psi \end{array} \quad \begin{array}{c} \vdots \\ \chi \end{array}}{\chi} \star 1$$

We can rewrite this subproof by moving the proof of $\phi \wedge \psi$ to the top of the proof of χ , and applying \wedge Elim to derive ϕ :

$$\frac{\begin{array}{c} \vdots \\ \phi \wedge \psi \end{array}}{\phi} \wedge \text{Elim} \quad \begin{array}{c} \vdots \\ \chi \end{array}$$

We can repeat this process for each application of $\star 1$ (starting with the smallest subproof) until we are left with a proof only using the original Natural Deduction rules.

Admissible rules 2

The rule $\star 2$ is not admissible. $P \rightarrow Q, \neg P \vdash \neg Q$ is not possible in unaugmented Natural Deduction, but is possible with $\star 2$:

$$\frac{\neg P \quad P \rightarrow Q}{\neg Q} \star 2$$

Admissible rules 3

The rule $\star 3$ is not admissible. $\vdash P$ is not possible in unaugmented Natural Deduction, but is possible with $\star 3$:

$$\frac{\frac{[\mathcal{Q}]}{Q \rightarrow Q} \rightarrow \text{Intro} \quad [(Q \rightarrow Q) \rightarrow P]}{P} \rightarrow \text{Elim}}{P} \star 3$$

Admissible rules 4

The rule $\star 4$ is admissible. We can rewrite any proof using $\star 4$ as a proof using \neg -Intro, \neg -Elim and \vee -Intro.

Suppose we have a proof involving one or more applications of $\star 4$. Each application corresponds to a subproof of the following form:

$$\begin{array}{c} [\neg\phi] \\ \vdots \\ \frac{\psi}{\phi \vee \psi} \star 4 \end{array}$$

We replace the $\star 4$ step itself with an application of \vee -Intro. At the top of the subproof we assume ϕ and $\neg(\phi \vee \psi)$ and apply \vee -Intro and \neg -Intro to derive $\neg\phi$. At the bottom of the subproof we assume $\neg(\phi \vee \psi)$ again and apply \neg -Elim to discharge assumptions of $\neg(\phi \vee \psi)$ and derive $\phi \vee \psi$.

The resultant subproof will look like this:

$$\begin{array}{c} \frac{\frac{[\phi]^1}{\phi \vee \psi} \vee\text{Intro} \quad [\neg(\phi \vee \psi)]^2}{\neg\phi} \neg\text{Intro}^1 \\ \vdots \\ \frac{\frac{\psi}{\phi \vee \psi} \vee\text{Intro} \quad [\neg(\phi \vee \psi)]^2}{\neg\phi} \neg\text{Elim}^2 \end{array}$$

This is a familiar proof structure: it is how we proved disjunctions such as $P \vee \neg P$ by indirect proof.

We can repeat this process for each application of $\star 4$ (starting with the smallest subproof) until we are left with a proof only using the original Natural Deduction rules.

Admissible rules 5

The rule $\star 5$ is admissible. We can rewrite any proof using $\star 5$ as a proof using \rightarrow -Intro and \neg -Elim.

Suppose we have a proof involving one or more applications of $\star 5$. Each application corresponds to a subproof of the following form:

$$\begin{array}{c} [\phi \rightarrow \psi] \\ \vdots \\ \frac{\phi}{\phi} \star 5 \end{array}$$

At the top of the subproof we assume $\neg\phi$ and ϕ and apply \neg Elim and \rightarrow Intro to derive $\phi \rightarrow \psi$. At the bottom of the subproof we assume $\neg\phi$ again and apply \neg Elim to discharge assumptions of $\neg\phi$ and derive ϕ .

The resultant subproof will look like this:

$$\frac{\frac{\frac{[\phi]^1 \quad [\neg\phi]^2}{\psi} \neg\text{Elim}}{\phi \rightarrow \psi} \rightarrow\text{Intro}^1}{\vdots} \quad \frac{\phi \quad [\neg\phi]^2}{\phi} \neg\text{Elim}^2}{\phi} \neg\text{Elim}^2$$

We can repeat this process for each application of $\star 5$ (starting with the smallest subproof) until we are left with a proof only using the original Natural Deduction rules.

Contraposition 1

$$\frac{\frac{\frac{[\neg P]^1 \quad [\neg\neg P]^2}{P} \neg\text{Elim}^1}{Q \rightarrow P} \rightarrow\text{Intro}}{\neg P} \neg(Q \rightarrow P) \neg\text{Elim}^2}{\neg P} \neg\text{Elim}^2$$

We cannot derive $\neg P$ by \neg Intro, where we assume P and derive a contradiction from it. Instead we must prove $\neg P$ indirectly, by assuming $\neg\neg P$ and deriving a contradiction from that. $\neg\neg P$ allows us to derive P (by assuming $\neg P$ and applying \neg Elim); from this we can derive $Q \rightarrow P$, which contradicts our premise $\neg(Q \rightarrow P)$.

Contraposition 2

$$\frac{\frac{\frac{[\neg(P \wedge \neg P)]^1 \quad [\neg\neg(P \wedge \neg P)]^3}{P \wedge \neg P} \wedge\text{Elim}}{P} \wedge\text{Elim}}{\neg(P \wedge \neg P)} \neg\text{Elim}^1 \quad \frac{\frac{[\neg(P \wedge \neg P)]^2 \quad [\neg\neg(P \wedge \neg P)]^3}{P \wedge \neg P} \wedge\text{Elim}}{\neg P} \wedge\text{Elim}}{\neg(P \wedge \neg P)} \neg\text{Elim}^2}{\neg(P \wedge \neg P)} \neg\text{Elim}^3$$

We proceed in a similar fashion to the previous question. We cannot derive $\neg(P \wedge \neg P)$ by assuming $P \wedge \neg P$ and deriving a contradiction from it; instead we must derive a contradiction from $\neg\neg(P \wedge \neg P)$. Assuming $\neg\neg(P \wedge \neg P)$ lets us derive $P \wedge \neg P$, which in turn lets us derive P and $\neg P$, and hence a contradiction.

Contraposition 3

First, we note that an application of \neg -Intro corresponds to a subproof of the following shape:

$$\frac{\begin{array}{c} [\phi] \quad [\phi] \\ \vdots \quad \vdots \\ \psi \quad \neg\psi \end{array}}{\neg\phi} \neg\text{Intro}$$

We replace all applications of \neg -Intro in our proof of ϕ from Γ (if there are any) with subproofs of the following shape:

$$\frac{\frac{[\neg\phi]^1 \quad [\neg\neg\phi]^3}{\phi} \neg\text{Elim}^1 \quad \frac{[\neg\phi]^2 \quad [\neg\neg\phi]^3}{\phi} \neg\text{Elim}^2}{\psi \quad \neg\psi} \neg\text{Elim}^3}{\neg\phi}$$

Contraposition 4

Suppose that $\Gamma \vdash_C \phi$. This means that there is a proof of ϕ from Γ which potentially uses contraposition (but doesn't use \neg -Intro or \neg -Elim).

If this proof involves no applications of contraposition, it is trivially true that $\Gamma \vdash \phi$.

If the proof does involve at least one application of contraposition, this application corresponds to a subproof of the following shape:

$$\frac{\begin{array}{c} [\neg\psi] \\ \vdots \\ \neg\phi \end{array}}{\phi \rightarrow \psi} \text{C}$$

We replace this subproof with a subproof of the following shape:

$$\frac{\frac{[\phi]^2 \quad \neg\phi}{\psi} \neg\text{Elim}^1}{\phi \rightarrow \psi} \rightarrow\text{Intro}^2}{\phi \rightarrow \psi}$$

We repeat this process for each application of contraposition, starting with the smallest subproof. The resultant proof has no applications of contraposition, but does have applications of \neg -Elim. Hence $\Gamma \vdash \phi$.

Contraposition 5

$$\frac{\neg\neg P}{\neg P \rightarrow \neg(Q \rightarrow Q)} \text{ C}$$

Contraposition 6

$$\frac{[\neg P] \quad \neg P \rightarrow \neg(Q \rightarrow Q)}{\neg(Q \rightarrow Q)} \rightarrow\text{Elim} \\ \frac{\neg(Q \rightarrow Q)}{(Q \rightarrow Q) \rightarrow P} \text{ C}$$

To show that $\neg P \rightarrow \neg(Q \rightarrow Q) \vdash_C (Q \rightarrow Q) \rightarrow P$ we are allowed to use $\rightarrow\text{Elim}$ as well as contraposition. To derive $(Q \rightarrow Q) \rightarrow P$ by contraposition, we need to provide a proof of $\neg(Q \rightarrow Q)$ from an assumption of $\neg P$. With $\neg P$ and our premise $\neg P \rightarrow \neg(Q \rightarrow Q)$ we can derive $\neg(Q \rightarrow Q)$ by $\rightarrow\text{Elim}$.

Contraposition 7

$$\frac{[\neg Q]}{Q \rightarrow Q} \text{ C} \\ \frac{Q \rightarrow Q \quad (Q \rightarrow Q) \rightarrow P}{P} \rightarrow\text{Elim}$$

Usefully, contraposition still allows us to easily derive $Q \rightarrow Q$, but instead of assuming and discharging Q (as we would in ordinary Natural Deduction), we assume and discharge $\neg Q$. We can then derive P by $\rightarrow\text{Elim}$ using our premise $(Q \rightarrow Q) \rightarrow P$.

Contraposition 8

$$\frac{[\neg Q]^1 \text{ C}^1 \quad \frac{[\neg P]^2 \quad \frac{\neg\neg P}{\neg P \rightarrow \neg(Q \rightarrow Q)} \text{ C}}{\neg(Q \rightarrow Q)} \rightarrow\text{Elim}}{(Q \rightarrow Q) \rightarrow P} \text{ C}^2}{P} \rightarrow\text{Elim}$$

The proof above combines our proofs from the previous three questions. This is not the shortest possible derivation of P from $\neg\neg P$; the proof below is shorter, but is perhaps less obvious:

$$\frac{\neg\neg P \quad \frac{[\neg P] \quad \frac{\neg\neg P}{\neg P \rightarrow \neg\neg P} \text{ C}}{\neg\neg P} \rightarrow\text{Elim}}{\neg\neg P \rightarrow P} \text{ C} \\ \frac{\neg\neg P \quad \neg\neg P \rightarrow P}{P} \rightarrow\text{Elim}$$

Contraposition 9

We note that each \neg -Elim step corresponds to a subproof of the following shape:

$$\frac{\begin{array}{c} [\neg\phi] \quad [\neg\phi] \\ \vdots \quad \vdots \\ \psi \quad \neg\psi \end{array}}{\phi} \neg\text{Elim}$$

We replace all applications of \neg -Elim with subproofs of the following shape:

$$\frac{\frac{\frac{[\neg P]^1}{P \rightarrow P} C^1}{\psi} \quad \frac{\frac{\frac{[\neg\phi]^2}{\psi \rightarrow \neg(P \rightarrow P)} C}{\neg(P \rightarrow P)} \neg\text{Elim}}{(P \rightarrow P) \rightarrow \phi} C^2}}{\phi} \neg\text{Elim}$$

Contraposition 10

Each application of \rightarrow -Intro corresponds to a subproof of the following shape:

$$\frac{\begin{array}{c} [\phi] \\ \vdots \\ \psi \end{array}}{\phi \rightarrow \psi} \rightarrow \text{Intro}$$

We replace each of these subproofs with a subproof of the following shape:

$$\frac{\frac{\frac{[\phi]^1}{\psi} \quad \frac{[\neg\psi]^2}{\neg\phi} \neg\text{Intro}^1}{\phi \rightarrow \psi} C^2}}{\phi \rightarrow \psi} C^2$$

Contraposition 11

To demonstrate that $\Gamma \vdash_{C'} \phi$ whenever $\Gamma \vdash \phi$ we need to be able to convert any proof in ordinary Natural Deduction to a proof with contraposition but without \rightarrow Intro, \neg Intro or \neg Elim.

We do this in three steps:

1. Replace \rightarrow Intro steps with \neg Intro and contraposition
2. Replace \neg Intro steps (including any new ones made in step 1) with \neg Elim
3. Replace \neg Elim steps (including any new ones made in step 2) with contraposition and \rightarrow Elim

This is by no means the only way of tackling this question, but given our answers to previous questions it is one of the easiest strategies. Question 10 showed us how to carry out step 1. Question 3 showed us how to carry out step 2. Finally, question 9 showed us how to carry out step 3.

After carrying out all three steps, we are left without any applications of \neg Intro, \neg Elim or \rightarrow Intro, but we may have applications of \rightarrow Elim and contraposition. This is a proof satisfying the requirements for $\Gamma \vdash_{C'} \phi$.

Hence if $\Gamma \vdash \phi$ then $\Gamma \vdash_{C'} \phi$.

This is a striking result: if we add the contraposition rule to Natural Deduction, we can dispense not only with \neg Intro but also with \neg Elim and \rightarrow Intro.

This not the only technique we could have used. For example, we could have replaced \rightarrow Intro steps with contraposition and \rightarrow Elim using the (monstrous) subproof below:

$$\begin{array}{c}
 \frac{\frac{\frac{[\neg\neg\phi]^4}{\neg\phi \rightarrow \neg(P \rightarrow P)} C \quad [\neg\phi]^3}{\neg(P \rightarrow P)} \rightarrow E}{\frac{[\neg P]^2}{P \rightarrow P} C^2 \quad \frac{\neg(P \rightarrow P)}{(P \rightarrow P) \rightarrow \phi} C^3}{\phi} \rightarrow E \\
 \vdots \\
 \frac{\psi}{\psi \rightarrow \neg(P \rightarrow P)} C \\
 \frac{\frac{[\neg P]^1}{P \rightarrow P} C^1 \quad \frac{\neg(P \rightarrow P)}{(P \rightarrow P) \rightarrow \neg\phi} C^4}{\frac{\neg\phi}{\phi \rightarrow \psi} C^5} \rightarrow E
 \end{array}$$