MORE EXERCISES

for the

Logic Manual

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Oxford

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Preface by Volker Halbach

This booklet contains additional exercises for the *Logic Manual* with solutions.

The student version of my own Exercises Booklet, which can be found here `http://logicmanual.philosophy.ox.ac.uk/exercises/exercises.pdf`, doesn't contain any solutions. Of course the reason for not making the solutions public was that, once you know the solutions are just a mouse click away, there is a temptation to look at them for some inspiration. So the solutions were made available only to tutors. However, I also realized that this isn't ideal. During revision one would like to be able to check whether an answer is correct.

Peter Fritz kindly agreed to write a new set of exercises with solutions. Peter and I hope that students will find them useful, especially when revising for an examination. The new exercises can also be used instead of the old Exercises Booklet in classes or tutorials. They resemble the old one in structure and difficulty.

When looking at the solution, one should bear in mind that they are only short hints. They should not be understood as perfect model solutions. In most cases good answers can differ significantly and the solutions usually contain only one possible answer. That doesn't mean that other answers are incorrect. This even applies to many formal questions. For instance, if there is a proof of a sentence in Natural Deduction at all, there are infinitely many proofs of the sentence. Some questions, that may look trivial, lead to deeper problems in philosophy of language and metaphysics. Peter has tried to keep the answers straightforward, but it shouldn't be assumed that they are the best solutions and philosophers may well differ on what the best solution is.

Some of the exercises are inspired by past examination papers set by James Studd, Gabriel Uzquiano and me. Peter and I thank the former two for allowing us to use their ideas here. Finally I would like to thank Peter for all the effort he has put into these pages.

Volker Halbach, January 2014

1 Sets, Relations and Arguments

Exercise 1.1. Consider the following sets:

(i) $\varnothing$
(ii) $\{$Mercury, Earth$\}$
(iii) $\{$Mercury, $\{$Earth$\}\}$
(iv) $\{$Mercury, Earth, $\varnothing\}$
(v) $\{$Mercury, Earth, Mercury$\}$
(vi) $\{$Mercury, Venus, Earth, Mars$\}$
(vii) $\{$Earth, the planet closest to the sun$\}$
(viii) $\{x : x$ is one of the four planets closest to the sun$\}$
(ix) $\{x : x$ is Neptune and one of the four planets closest to the sun$\}$
(x) the set containing Mercury, Earth, and the four planets closest to the sun

Which of the indices (i)–(x) list the same set?

Exercise 1.2. Consider the following relation R:

$$\{\langle \text{Earth, Venus}\rangle, \langle \text{Venus, Earth}\rangle, \langle \text{Venus, Mars}\rangle, \langle \text{Mars, Mars}\rangle\}$$

(a) Draw a diagram of R.
(b) Determine whether R is reflexive on $\{$Venus, Earth, Mars$\}$ and whether it is symmetric.
(c) Let S be the smallest transitive relation containing all elements of R. Write down S as R is written down above, and draw a diagram of S.
(d) Determine whether S is reflexive on $\{$Venus, Earth, Mars$\}$ and whether it is symmetric.

Exercise 1.3. Consider the following relations:

(i) the relation containing all pairs of persons $\langle d, e\rangle$ such that e is d's mother,
(ii) the relation containing all pairs of persons $\langle d, e\rangle$ such that e and d have the same mother,
(iii) the relation containing all pairs of persons $\langle d, e\rangle$ such that e is distinct from d.

Let S be the set of all persons. Determine for each of the relations (i)–(iii)

(a) whether it is reflexive on S,
(b) whether it is symmetric,
(c) whether it is asymmetric,
(d) whether it is antisymmetric,
(e) whether it is transitive,
(f) whether it is a function,
(g) whether it is an equivalence relation on S.

Exercise 1.4. Consider the following relation R:

$$\{\langle 2,1\rangle, \langle 3,10\rangle, \langle 4,2\rangle, \langle 5,16\rangle, \langle 6,3\rangle, \langle 7,22\rangle\}.$$

Check for yourself that R is a function.

(a) Specify the domain and range of R by enumerating their elements.
(b) Specify the set S of sets M such that R is a function into M without reference to R.

Exercise 1.5. A set A is a subset of a set B if and only if all elements of A are also elements of B. Establish the following claims:

(a) There is a set which is the subset of the empty set.
(b) If A is a subset of B and B is a subset of A, then A is B.
(c) For any relation R, any set B and any subset A of B, if R is transitive on B, it is also transitive on A.
(d) There is some relation R, some set B and some subset A of B such that R is transitive on A but not transitive on B.
(e) There is a relation R, subset S of R and set A such that R is transitive on A but S is not transitive on A.

(f) There is a relation R, subset S of R and set A such that S is transitive on A but R is not transitive on A.

Exercise 1.6. (a) Show that every asymmetric relation is antisymmetric.
(b) Show there is an antisymmetric relation which is not asymmetric.
(c) Show that there is a relation which is neither symmetric nor asymmetric nor antisymmetric.
(d) Show that the empty set is a relation which is symmetric, asymmetric and antisymmetric. Show that it is the only relation which is both symmetric and asymmetric.
(e) Show that there is a relation which is symmetric, antisymmetric and not empty.
(f) Show that there is a relation which is asymmetric, antisymmetric and not empty.

Exercise 1.7. Consider any function R. Let S be the relation which holds between elements of the domain of R if and only if they are mapped to the same element by R. In other words, let S be the set of pairs $\langle d, e \rangle$ for which there is an f such that R contains both $\langle d, f \rangle$ and $\langle e, f \rangle$. Show that S is an equivalence relation on the domain of R.

Exercise 1.8. Identify the premises and conclusions in the following arguments and determine whether they are logically valid.

(i) Since the sky is blue, the sky is colored.
(ii) All reptiles are mammals. No crocodile is a mammal. Therefore no crocodile is a reptile.
(iii) Paula is cold; this is because she shivers and if someone is cold, they shiver.
(iv) There are no mammals which lay eggs. It follows that humans are not mammals, since the platypus is a mammal which lays eggs.
(v) All birds fly and no birds fly. Therefore there are no birds.
(vi) There are no animals, since all fish swim and no fish swim.

Exercise 1.9. (a) Show, in your own words, that an argument is valid if and only if the set obtained by adding the negation of the conclusion to the premises is inconsistent.

(b) Show that two logically equivalent sentences are either both logical truths or both not logical truths.

Answers for Chapter 1

Answer to 1.1. Since Neptune is not one of the four planets closest to the sun, nothing is both Neptune and one of the four planets closest to the sun. Therefore $\varnothing = \{x : x \text{ is Neptune and one of the four planets closest to the sun}\}$, so (i) and (ix) list the same set.

{Mercury, Earth} = {Mercury, Earth, Mercury,}, and as Mercury is the planet closest to the sun, also {Mercury, Earth} = {Earth, the planet closest to the sun}. So (ii), (v) and (vii) list the same set. Note that {Earth} (i.e., the set containing only Earth) is not identical to Earth, so {Mercury, {Earth}} is distinct from {Mercury, Earth}. {Mercury, Earth, $\varnothing$} is also distinct from {Mercury, Earth} since only the former contains $\varnothing$ as an element.

Mercury, Venus, Earth and Mars are the four planets closest to the sun, so {Mercury, Venus, Earth, Mars} is identical to $\{x : x$ is one of the four planets closest to the sun$\}$ as well as the set containing Mercury, Earth, and the four planets closest to the sun. So (vi), (viii) and (x) list the same set.

Answer to 1.2.

(a)

$$\begin{tikzcd} \text{Earth} \arrow[r, bend left] & \text{Venus} \arrow[l, bend left] \arrow[d] \\ & \text{Mars} \arrow[loop below] \end{tikzcd}$$

(b) R is not reflexive on {Venus, Earth, Mars} as it contains neither ⟨Earth, Earth⟩ nor ⟨Venus, Venus⟩. R is not symmetric as it contains ⟨Venus, Mars⟩ but not ⟨Mars, Venus⟩.

(c) Any transitive relation containing all elements of R must contain ⟨Earth, Mars⟩, ⟨Earth, Earth⟩, ⟨Venus, Venus⟩. The set containing these pairs and those in R is a transitive relation, hence S is {⟨Earth, Venus⟩, ⟨Venus, Earth⟩, ⟨Venus, Mars⟩, ⟨Mars, Mars⟩, ⟨Earth, Mars⟩, ⟨Earth, Earth⟩, ⟨Venus, Venus⟩}.

$$\begin{tikzcd}
\text{Earth} \arrow[loop above] \arrow[r, bend left] \arrow[dr] & \text{Venus} \arrow[loop above] \arrow[l, bend left] \arrow[d] \\
& \text{Mars} \arrow[loop below]
\end{tikzcd}$$

(d) S is reflexive. As in (b), S is not symmetric.

Answer to 1.3. (i) is not reflexive on S since some person is not their own mother. (i) is not symmetric since some person is not their mother's mother. (i) is asymmetric since no person is their mother's mother. (i) is antisymmetric since, as just noted, there are no persons d and e such that e is d's mother and d is e's mother, and therefore trivially, for any such persons, d is e. (i) is not transitive since there is a person such that their mother's mother is not their mother. (i) is a function since every person has no more than one mother. (i) is not an equivalence relation S since it is not reflexive on S.

(ii) is reflexive on S since every person has the same mother as themselves. (ii) is symmetric since if e and d have the same mother, d and e have the same mother. (ii) is not asymmetric since there are persons d and e who have the same mother. (ii) is not antisymmetric since there are *distinct* persons d and e who have the same mother. (ii) is transitive since if d and e have the same mother and e and f have the same mother, then e and f have the same mother. (ii) is not a function since for some person d, there are two distinct persons e and e' such that d and e and d and e' have the same mother. (ii) is an equivalence relation on S, since, as we have argued, (ii) is reflexive on S, transitive and symmetric.

(iii) is not reflexive on S since no person is distinct from themselves. (iii) is symmetric since if one person is distinct from another, then the latter is distinct from the former. (iii) is not asymmetric since there are persons d and e who are distinct from another. (iii) is not antisymmetric since there are *distinct* persons d and e who are distinct from another. (iii) is not transitive since there are persons d and e such that d is distinct from e and e is distinct from d, but of course d is not distinct from d. (iii) is not a function since there is a person who is distinct from two distinct persons. (iii) is not an equivalence relation on S since it is not reflexive on S.

Answer to 1.4.

(a) The domain of R is the set $\{2, 3, 4, 5, 6, 7\}$. The range of R is the set $\{1, 2, 3, 10, 16, 22\}$.

(b) S is the set $\{M : M$ is a set such that 1, 2, 3, 10, 16, and 22 are elements of $M\}$.

Answer to 1.5.

(a) All elements of the empty set are elements of the empty set, therefore the empty set is a subset of the empty set.

(b) If A is a subset of B and B is a subset of A, then every element of A is an element of B and every element of B is an element of A. Hence such sets A and B contain the same members and therefore are identical.

(c) Let e, d, f be elements of A such that R contains $\langle d, e\rangle$ and $\langle e, f\rangle$. Since A is a subset of B, e, d, f are also elements of B. R is transitive on B, therefore it must contain $\langle d, f\rangle$. Hence R is transitive on A.

(d) Let $R = \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$, $B = \{0, 1, 2\}$ and $A = \varnothing$. Since A is the empty set it is a subset of B and transitive on A. But as R does not contain $\langle 0, 2\rangle$, R is not transitive on B.

(e) Let $R = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 0, 2\rangle\}$, $S = \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$ and $A = \{0, 1, 2\}$. R is transitive on A, but as in (c), S is not transitive on A.

(f) Let $R = \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$, $S = \varnothing$ and $A = \{0, 1, 2\}$. S is transitive on A, but as in (c), R is not transitive on A.

Answer to 1.6.

(a) Let R be an asymmetric relation. For R not to be antisymmetric, there would have to be distinct d and e such that $\langle d, e\rangle$ and $\langle e, d\rangle$. But since R is asymmetric, there are no such elements, hence R is also antisymmetric.

(b) $R = \{\langle 0, 0\rangle\}$ is antisymmetric but not asymmetric.

(c) Let $R = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle\}$. R is not symmetric since it contains $\langle 0, 1\rangle$ but not $\langle 1, 0\rangle$. R is neither asymmetric nor antisymmetric since it contains both $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$.

(d) Since the empty set contains no pairs $\langle d, e\rangle$, it trivially satisfies the conditions for being symmetric, asymmetric and antisymmetric. Consider any relation R distinct from the empty set. Since it is not empty, R contains some pair $\langle d, e\rangle$. If R is asymmetric, it does not contain $\langle e, d\rangle$ as well, hence it is not symmetric. So R is not both symmetric and antisymmetric. Thus $\varnothing$ is the only relation which is both symmetric and asymmetric.

(e) $R = \{\langle 0, 0\rangle\}$ is symmetric, antisymmetric and not empty.

(f) $R = \{\langle 0, 1\rangle\}$ is asymmetric, antisymmetric and not empty.

Answer to 1.7. We first prove that S is reflexive on the domain of R. For any d in the domain of R, there is an e such that R contains $\langle d, e\rangle$, and so S contains $\langle d, d\rangle$. Therefore S is reflexive on the domain of R.

For symmetry, consider any d and e such that $\langle d, e\rangle \in S$. Then there is an f such that R contains both $\langle d, f\rangle$ and $\langle e, f\rangle$. So R contains $\langle e, f\rangle$ and $\langle d, f\rangle$, and therefore $\langle e, d\rangle \in S$. Therefore S is symmetric.

For transitivity, consider any d, e and f such that $\langle d, e\rangle \in S$ and $\langle e, f\rangle \in S$. Then there are g and h such that R contains $\langle d, g\rangle$, $\langle e, g\rangle$, $\langle e, h\rangle$ and $\langle f, h\rangle$. Since F is a function, it follows from $\langle e, g\rangle$ and $\langle e, h\rangle$ that $g = h$. Hence there is an i (namely g, which is identical to h) such that R contains both $\langle d, i\rangle$ and $\langle f, i\rangle$, and so $\langle d, f\rangle \in S$. Therefore S is transitive.

Answer to 1.8. I only identify the conclusion of each argument.

(i) 'The sky is colored'. This argument is not logically valid.
(ii) 'No crocodile is a reptile'. This argument is logically valid.
(iii) 'Paula is cold'. This argument is not logically valid.
(iv) 'Humans are not animals'. This argument is logically valid (note that the set of premises is inconsistent).
(v) 'There are no birds'. This argument is logically valid.
(vi) 'There are no animals'. This argument is not logically valid.

Answer to 1.9.

(a) An argument is valid if and only if there is no interpretation under which the premises are all true and the conclusion is false. Under any interpretation, the conclusion is false if and only if its negation is true. Hence an argument is logically valid if and only if there is no interpretation under which the premises and the negation of the conclusion are all true. So an argument is logically valid if and only if there is no interpretation under which all sentences in the set obtained by adding the negation of the conclusion to the premises are true; this is the case if and only if this set is inconsistent.
(b) Let A and B be logically equivalent sentences. A is logically true if and only if A is true under every interpretation. Since A and B are logically equivalent, under every interpretation, A is true if and only if B is true. So A is logically true if and only if B is true under every interpretation, which is the case if and only if B is logically true.

2 Syntax and Semantics of Propositional Logic

Exercise 2.1. Describe the different ways in which quotation marks may be added to the following expressions to obtain *true* English sentences.

(i) is a closing quotation mark.
(ii) dfswse is an English sentence is obviously an incorrect claim or just nonsense.
(iii) is identical with
(iv) Achilles denotes Achilles, which in turn denotes Achilles, which is an expression but not a person.

Exercise 2.2. Establish the following claims using truth tables. You may use partial truth tables.

(i) $((P \to Q) \to P) \to P$ is a tautology.
(ii) $((P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow R)) \leftrightarrow (Q \leftrightarrow R)$ is a tautology.
(iii) $P \lor Q, \neg P \vDash Q$
(iv) $P \to Q, Q \to R \vDash P \to R$
(v) $P \to (Q \to R), P \to Q \vDash P \to R$

Exercise 2.3. Classify the following sentences as tautologies, contradictions or as sentences which are neither.

(i) $((P_1 \to P_2) \to P_3) \to ((P_3 \to P_1) \to \neg(P_4 \to P_1))$
(ii) $((P_1 \to P_2) \to P_3) \to ((P_3 \to P_1) \to (\neg P_4 \to P_1))$
(iii) $((P_1 \to P_2) \to P_3) \to ((P_3 \to P_1) \to (P_4 \to \neg P_1))$

Exercise 2.4. Similar to the connective $\downarrow$ representing 'neither … nor …', we can add to $\mathcal{L}_1$ a connective $\uparrow$ representing 'not both … and …'. This is called the

Sheffer stroke, and can be pronounced ‘nand’. As we can define $\phi \downarrow \psi$ in $\mathcal{L}_1$ as $\neg\phi \wedge \neg\psi$, we can define $\phi \uparrow \psi$ in $\mathcal{L}_1$ as $\neg(\phi \wedge \psi)$.

(a) Write down the truth table for $\uparrow$.
(b) Find a formula of $\mathcal{L}_1$ which defines $\phi \uparrow \psi$ and only contains the connectives $\neg$ and $\vee$.
(c) Show that $(P \uparrow P) \uparrow P$ is a tautology.[1]
(d) Show that $P, P \uparrow (Q \uparrow R) \vDash R$.
(e) Just as we can define $\uparrow$ using $\neg$ and $\wedge$, we can define $\neg$ and $\wedge$ using $\uparrow$. Show that this is the case; i.e., find a propositional sentence containing only the connective $\uparrow$ which is logically equivalent to $\neg\phi$, and a propositional sentence containing only the connective $\uparrow$ which is logically equivalent to $\phi \wedge \psi$.
(f) Use the claims established in (e) to argue that the language of propositional sentences containing only $\uparrow$ is truth-functionally complete.

Exercise 2.5. Consider the following relations:

(i) R_1 is the relation containing exactly the pairs $\langle \phi, \psi \rangle$ of sentences of $\mathcal{L}_1$ such that $\phi \wedge \psi$ is a tautology,
(ii) R_2 is the relation containing exactly the pairs $\langle \phi, \psi \rangle$ of sentences of $\mathcal{L}_1$ such that $\{\phi, \psi\}$ is semantically consistent,
(iii) R_3 is the relation containing exactly the pairs $\langle \phi, \psi \rangle$ of sentences of $\mathcal{L}_1$ such that $\phi \vDash \psi$,
(iv) R_4 is the relation containing exactly the pairs $\langle \phi, \psi \rangle$ of sentences of $\mathcal{L}_1$ such that ϕ and ψ are logically equivalent.

Determine for each of these relations

(a) whether it is reflexive on the set of sentences of $\mathcal{L}_1$,
(b) whether it is symmetric,
(c) whether it is transitive.

1 Strictly speaking, we have not defined what it means for a sentence including $\uparrow$ to be a tautology, since the notion of a tautology is only defined for sentences of $\mathcal{L}_1$, which does not include $\uparrow$. But given the truth-table of $\uparrow$, it requires no changes to the definition of a tautology to apply it to sentences involving $\uparrow$ as well, and this is what is intended in this exercise. Similarly remarks apply to (d) and (e).

Exercise 2.6. Classify each of the following sets as semantically consistent or inconsistent.

(i) $\{\phi : \phi$ is a sentence letter$\}$
(ii) $\{\neg\phi : \phi$ is a sentence letter$\}$
(iii) $\{\phi : \phi$ is a sentence of $\mathcal{L}_1$ not containing $\neg\}$

Exercise 2.7. Let ϕ and ψ be sentences of $\mathcal{L}_1$. Determine which of the following claims are correct.

(i) ϕ is a tautology if and only if $\neg\phi$ is a contradiction.
(ii) ϕ and ψ are logically equivalent if and only if $\phi \vDash \psi$ and $\psi \vDash \phi$.
(iii) If ϕ is a tautology if and only if ψ is a tautology then ϕ and ψ are logically equivalent.
(iv) If ϕ and ψ are logically equivalent then ϕ is a tautology if and only if ψ is a tautology.

Exercise 2.8. Show that Theorem 2.14 is correct, i.e., that for any sentences ψ_1, $\ldots$, ψ_n and ϕ of $\mathcal{L}_1$, $\psi_1, \ldots, \psi_n \vDash \phi$ if and only if $\psi_1 \wedge \cdots \wedge \psi_n \to \phi$ is a tautology.

Exercise 2.9. Let ϕ be a sentence of $\mathcal{L}_1$ and Γ and Δ be sets of sentences of $\mathcal{L}_1$. We write $\Gamma \cup \Delta$ for the set of sentences which are contained in at least one of the sets Γ and Δ; this is called the union of Γ and Δ. Show that the following claims are correct.

(i) $\phi \vDash \phi$
(ii) If $\Gamma \vDash \phi$ then $\Gamma \cup \Delta \vDash \phi$.
(iii) If $\Gamma \vDash \phi$ and $\Delta \vDash \psi$ for all $\psi \in \Gamma$ then $\Delta \vDash \phi$.

Answers for Chapter 2

Answer to 2.1.

(i) ‘’’ is a closing quotation mark.

(ii) There are infinitely many solutions. Two natural solutions are:

> ‘dfswse is an English sentence’ is obviously an incorrect claim or just nonsense.
>
> ‘‘dfswse’ is an English sentence’ is obviously an incorrect claim or just nonsense.

But any way of inserting quotation marks in ‘dfswse is an English sentence’ produces an obviously incorrect claim or just nonsense, hence any such way yields a true English sentence when applied to the relevant part of (ii).

(iii) Again, there are infinitely many solutions, since for any string σ of quotation marks – opening or closing – the following is a true English sentence:

> ‘σ’ is identical with ‘σ’

(iv) There are also infinitely many solutions to this exercise. For any strings of quotation marks σ_1 and σ_2, the following is a true English sentence:

> ‘‘‘σ_1Achillesσ_2’’’ denotes ‘‘σ_1Achillesσ_2’’, which in turn denotes ‘σ_1Achillesσ_2’, which is an expression but not a person.

Answer to 2.2. Solutions omitted.

Answer to 2.3. (i) is a contradiction and (ii) a tautology; this can be established using partial truth tables. (iii) is neither a tautology nor a contradiction; this is because (iii) is true in the $\mathcal{L}_1$-structure which assigns T to every sentence letter, and (iii) is false in any $\mathcal{L}_1$-structure which assigns F to P_1 and P_3.

Answer to 2.4.

(a) The truth table for $\uparrow$ is:

ϕ	ψ	$\phi \uparrow \psi$
T	T	F
T	F	T
F	T	T
F	F	T

(b) $\neg\phi \vee \neg\psi$ defines $\phi \uparrow \psi$.

(c) This can be established using a partial truth table.

(d) Likewise.

(e) $\phi \uparrow \phi$ is logically equivalent to $\neg\phi$.
$(\phi \uparrow \psi) \uparrow (\phi \uparrow \psi)$ is logically equivalent to $\phi \wedge \psi$.

(f) Recall that all truth tables can be produced using propositional sentences containing only $\neg$ and $\wedge$. In (e) we have established that both $\neg\phi$ and $\phi \wedge \psi$ can be defined using $\uparrow$. Hence every truth table can be produced using propositional sentences containing only $\uparrow$. So the language of propositional sentences containing only $\uparrow$ is truth-functionally complete.

Answer to 2.5. R_1 is not reflexive on the set of sentences of $\mathcal{L}_1$ since $P \wedge P$ is not a tautology. R_1 is symmetric and transitive.

R_2 is not reflexive on the set of sentences of $\mathcal{L}_1$ since $\{P \wedge \neg P, P \wedge \neg P\}$ is not semantically consistent. R_2 is symmetric. R_2 is not transitive, since $\{P, Q\}$ and $\{Q, \neg P\}$ are both semantically consistent but $\{P, \neg P\}$ is not semantically consistent.

R_3 is reflexive on the set of sentences of $\mathcal{L}_1$. R_3 is not symmetric since $P \vDash P \vee Q$ but not $P \vee Q \vDash P$. R_3 is transitive.

R_4 is reflexive on the set of sentences of $\mathcal{L}_1$, symmetric and transitive.

Answer to 2.6. All three sets are semantically consistent:

The sentences of (i) are all true in the $\mathcal{L}_1$-structure which assigns T to every sentence letter.

The sentences of (ii) are all true in the $\mathcal{L}_1$-structure which assigns F to every sentence letter.

The sentences of (iii) are all true in the $\mathcal{L}_1$-structure which assigns T to every sentence letter. To demonstrate this last claim, note if ϕ and ψ are true in an $\mathcal{L}_1$-structure, then $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$ and $\phi \leftrightarrow \psi$ are all true in this structure as well. Thus a sentence built up using only these connectives is true in an $\mathcal{L}_1$-structure given that all of the sentence letters occurring in it are true in this structure. Since the only connective besides $\wedge$, $\vee$, $\to$ and $\leftrightarrow$ in $\mathcal{L}_1$ is $\neg$, it follows that any sentence of $\mathcal{L}_1$ not containing $\neg$ is true in the $\mathcal{L}_1$-structure which assigns T to every sentence letter.

Answer to 2.7. (i) is correct: ϕ is a tautology if and only if ϕ is true in every $\mathcal{L}_1$-structure. In any $\mathcal{L}_1$-structure, ϕ is true if and only if $\neg\phi$ is false, so ϕ is true in every $\mathcal{L}_1$-structure if and only if $\neg\phi$ is false in every $\mathcal{L}_1$-structure. $\neg\phi$ is a contradiction if and only if $\neg\phi$ is false in every $\mathcal{L}_1$-structure. Therefore ϕ is a tautology if and only if $\neg\phi$ is a contradiction.

(ii) is correct: $\phi \vDash \psi$ if and only if ψ is true in every $\mathcal{L}_1$-structure in which ϕ is true, and $\psi \vDash \phi$ if and only if ϕ is true in every $\mathcal{L}_1$-structure in which ψ is true. So $\phi \vDash \psi$ and $\psi \vDash \phi$ if and only if ϕ and ψ are true in exactly the same $\mathcal{L}_1$-structures, which is the case if and only if ϕ and ψ are logically equivalent.

(iii) is incorrect: neither P nor Q is a tautology, so P is a tautology if and only if Q is a tautology, but P and Q are not logically equivalent.

(iv) is correct: if ϕ and ψ are logically equivalent, then ϕ and ψ are true in exactly the same $\mathcal{L}_1$-structures, so ϕ is true in all $\mathcal{L}_1$-structures if and only if ψ is true in all $\mathcal{L}_1$-structures; therefore, ϕ is a tautology if and only if ψ is a tautology.

Answer to 2.8. $\psi_1, \dots, \psi_n \vDash \phi$ if and only if there is no $\mathcal{L}_1$-structure in which all of $\psi_1, \dots, \psi_n$ are true but ϕ is false. In any $\mathcal{L}_1$-structure, all of $\psi_1, \dots, \psi_n$ are true if and only if $\psi_1 \wedge \dots \wedge \psi_n$ is true. Further, in any $\mathcal{L}_1$-structure, $\psi_1 \wedge \dots \wedge \psi_n$ is true and ϕ is false if and only if $\psi_1 \wedge \dots \wedge \psi_n \to \phi$ is false. So $\psi_1, \dots, \psi_n \vDash \phi$ if and only if there is no $\mathcal{L}_1$-structure in which $\psi_1 \wedge \dots \wedge \psi_n \to \phi$ is false, which in turn is the case if and only if $\psi_1 \wedge \dots \wedge \psi_n \to \phi$ is a tautology.

Answer to 2.9. (i) There is no $\mathcal{L}_1$-structure in which ϕ is true and ϕ is false, so $\phi \vDash \phi$.

(ii) If $\Gamma \vDash \phi$ then ϕ is true in all $\mathcal{L}_1$-structures in which all sentences in Γ are true. Since all sentences in Γ are in $\Gamma \cup \Delta$, every $\mathcal{L}_1$-structure in which all sentences of $\Gamma \cup \Delta$ are true is an $\mathcal{L}_1$-structure in which all sentences of Γ are true. So every $\mathcal{L}_1$-structure in which all sentences of $\Gamma \cup \Delta$ are true is an $\mathcal{L}_1$-structure in which ϕ is true, and therefore $\Gamma \cup \Delta \vDash \phi$.

(iii) If $\Delta \vDash \psi$ for all $\psi \in \Gamma$, then in every $\mathcal{L}_1$-structure in which all sentences in Δ are true, all sentences of Γ are true. If also $\Gamma \vDash \phi$, then in every $\mathcal{L}_1$-structure in which all sentences in Γ are true, ϕ is true as well. Thus in every $\mathcal{L}_1$-structure in which all sentences in Δ are true, ϕ is true as well, and therefore $\Delta \vDash \phi$.

3 Formalization in Propositional Logic

Exercise 3.1. Which of the following connectives are truth-functional? Draw truth tables for each of them, indicating failures of truth-functionality using '?'.

(i) It's possible that *A*.
(ii) It's not necessary that *A*, but it's not possible that *A* is not the case.
(iii) Neither *A* nor *B*.
(iv) Some Roman knew that *A*.
(v) The Romans knew that *A* because the Greeks knew that *B*.
(vi) A few Romans knew that *A* because Jupiter told them.

Exercise 3.2. Discuss whether the following argument is valid. What does this show us about using the connective $\rightarrow$ to formalize 'if' in English?

> If the accused didn't commit the crime, then someone else did; therefore if the accused hadn't committed the crime, then someone else would have.

Exercise 3.3. Formalize the following two arguments as valid arguments in $\mathcal{L}_1$, rewording the premisses as necessary. Demonstrate the validity of the arguments using full or partial truth tables.

(i) If both Alice and Barbara admit to having hacked into government computers, then neither of them will receive a prison sentence. But if either of them admits to having hacked into a computer while the other doesn't, she will be sentenced to imprisonment while the other won't. So unless both don't admit the deed, it cannot happen that both receive a prison sentence.

(ii) God is omnibenevolent. It follows that causation is non-transitive. For supposing causation to be transitive, given that God caused the existence and flourishing of humankind, it must also be the case that God caused global climate change if humankind's existence and flourishing did. Every honest scientist knows that exactly this human activity did cause our present climate predicament. But although every good Christian knows that God created humankind and caused it to flourish, a God behind global climate change is far from omnibenevolent.

Exercise 3.4. Determine the scopes of the underlined occurrences of connectives in the following sentences, which have been abbreviated in accordance with the bracketing conventions.

(i) $P \land P \underline{\land} P \land P$
(ii) $P \underline{\leftrightarrow} Q \lor \neg R$
(iii) $R \to ((P \lor Q) \land \neg P \underline{\to} Q)$
(iv) $P \land (Q \lor R) \underline{\land} P_1 \to R_1 \land R_2$

Exercise 3.5. Consider the following sentences:

(i) If Alice is happy she forgot Bob's birthday and she is at a party.
(ii) Alice forgot Bob's birthday and is at a party or is not happy.

Both of them are ambiguous.

(a) Determine all possible readings of both sentences.
(b) Formalize all readings of both sentences in the language of propositional logic, using the same dictionary.
(c) The different formalizations of (i) and (ii) give rise to several formalizations of the argument whose single premise is (i) and whose conclusion is (ii). Determine which of these are valid.

Exercise 3.6. Formalize the following sentences in the language of propositional logic. Your formalizations should be as detailed as possible. If there are several equally natural formalizations – e.g., if the sentence is ambiguous – list all of them and describe the differences.

(i) 105 is divisible by 3, 5 and 7.
(ii) 15 is the sum of 3, 5 and 7.
(iii) Alice married and got pregnant.
(iv) Anne and Barbara carried the piano and sweated.

Exercise 3.7. Consider the following arguments:

(i) Antonia is a mammal. If Antonia is a mammal, she is mortal. Therefore Antonia is mortal.
(ii) Antonia is a mammal. Every mammal is mortal. Therefore Antonia is mortal.
(iii) Antonia is a mammal. Antonia is not a mammal. Therefore Antonia is mortal.
(iv) It is not the case that if Antonia is a mammal, she is mortal. Antonia is not a mammal. Therefore Antonia is mortal.

For each of these arguments

(a) formalize it in propositional logic,
(b) determine whether it is propositionally valid,
(c) determine whether it is valid.

Answers for Chapter 3

Answer to 3.1.

(i) Not truth-functional. The sentences 'There are only finitely many primes' and 'There are no donkeys' are both false, but it is possible that there are no donkeys and not possible that there are finitely many primes.

A	It's possible that A.
T	T
F	?

(ii) Truth-functional. It is necessary that A if and only if it is not possible that A is not the case, so it cannot be the case that it is both not necessary that A and not possible that A is not the case. Hence such a compound sentence is false for every sentence A.

A	It's not necessary that A, but it's not possible that A is not the case.
T	F
F	F

(iii) Truth-functional. 'Neither A nor B' is true if and only if both A and B are false.

A	B	Neither A nor B.
T	T	F
T	F	F
F	T	F
F	F	T

(iv) Not truth-functional. The sentences '2+2=4' and 'π is irrational' are both true, but some Roman knew that 2+2=4 and no Roman knew that π is irrational.

A	Some Roman knew that A.
T	?
F	F

(v) Not truth-functional. The Romans learned much from the Greeks, so there are some true sentences A and B such that 'The Romans knew that A because the Greeks knew that B' is true. But 'π is irrational' is true as well, and it is not the case that the Romans knew that π is irrational because the Greeks knew that π is irrational.

A	B	The Romans knew that A because the Greeks knew that B.
T	T	?
T	F	F
F	T	F
F	F	F

(vi) Truth-functional. There is no Jupiter, so Jupiter did not tell any Roman anything, and therefore for every sentence A, 'A few Romans knew that A because Jupiter told them' is false.

A	A few Romans knew that A because Jupiter told them.
T	F
F	F

Answer to 3.2. This argument is not valid according to Characterization 1.9 of logical validity. One of many ways of showing that this is the case is the following: Consider the interpretation on which 'the accused' is interpreted as 'Oswald' and 'crime' as 'murder of JFK'. If Oswald didn't commit the murder of JFK, then someone else did, but it is not the case that if Oswald hadn't committed the murder of JFK, then someone else would have. So there is an interpretation of the argument under which the premise is true and the conclusion false, and so the argument is not valid.

If we were to formalize the argument using $\neg P \to Q$ for the premise as well as the conclusion, where

P: The accused committed the crime.
Q: Someone other than the accused committed the crime.

then the argument would be propositionally valid, and therefore valid. But as argued above, the argument is not valid, so this formalization is not adequate. The culprit is the conclusion; a counterfactual conditional can be false even though its antecedent is false. The example therefore shows that even though counterfactual conditionals are constructed in English using 'if' (and sometimes 'then'), they can not be formalized using $\to$.

Answer to 3.3. (i) The argument can be formalized as the following valid argument in $\mathcal{L}_1$:

$$P_1 \land P_2 \to \neg Q_1 \land \neg Q_2,$$
$$\neg(P_1 \leftrightarrow P_2) \to (P_1 \leftrightarrow Q_1) \land (P_2 \leftrightarrow Q_2)$$
$$\vDash Q_1 \land Q_2 \to \neg P_1 \land \neg P_2$$

P_1: Alice admits to having hacked into government computers.
P_2: Barbara admits to having hacked into government computers.
Q_1: Alice will receive a prison sentence.
Q_2: Barbara will receive a prison sentence.

Truth table omitted.

(ii) The argument can be formalized as the following valid argument in $\mathcal{L}_1$:

$$P_1, Q \to (P_2 \to (R \to P_3)), R, P_2, P_3 \to \neg P_1 \vDash \neg Q$$

P_1: God is omnibenevolent.
P_2: God caused the existence and flourishing of humankind.
P_3: God caused global climate change.
Q: Causation is transitive.
R: Humankind's existence and flourishing caused global climate change.

Truth table omitted.

Answer to 3.4. I have underbraced the respective scopes, and added the brackets that have been dropped in accordance with the bracketing conventions.

(i) $(\underbrace{((P \wedge P) \underline{\wedge} P)}_{} \wedge P)$

(ii) $\underbrace{(P \underline{\leftrightarrow} (Q \vee \neg R))}_{}$

(iii) $(R \to \underbrace{(((P \vee Q) \wedge \neg P) \underline{\to} Q)}_{})$

(iv) $(\underbrace{((P \wedge (Q \vee R)) \underline{\wedge} P_1)}_{} \to (R_1 \wedge R_2))$

Answer to 3.5. (a) There are two readings of (i). According to the first, the consequent of the conditional is 'she forgot Bob's birthday'; according to the second, it is 'she forgot Bob's birthday and she is at a party'. There are also two readings of (ii). According to the first, the first disjunct is 'Alice forgot Bob's birthday and is at a party'; according to the second, the second conjunct is '[Bob] is at a party or is not happy'.

(b) I will use the following dictionary:

P: Alice is happy.
Q: Alice forgot Bob's birthday.
R: Alice is at a party.

With this, we can formalize the two readings of (i) using (i1) and (i2), and the two readings of (ii) using (ii1) and (ii2):
(i1) $(P \to Q) \wedge R$
(i2) $P \to (Q \wedge R)$
(ii1) $(Q \wedge R) \vee \neg P$
(ii2) $Q \wedge (R \vee \neg P)$

(c) (ii1) follows logically from (i1), and (ii1) follows logically from (i2). This can be shown using partial truth tables. (ii2) does not follow logically from (i1), and (ii2) does not follow logically from (i2): in any $\mathcal{L}_1$-structure which assigns F to P and Q and T to R, both (i1) and (i2) are true and (ii2) is false.

Answer to 3.6. (i) $P \wedge Q \wedge R$

P: 105 is divisible by 3.
Q: 105 is divisible by 5.
R: 105 is divisible by 7.

(ii) P

P: 15 is the sum of 3, 5 and 7.

(iii) In some circumstances, it is natural to formalize this as follows: $P \wedge Q$, where

P: Alice married.
Q: Alice got pregnant.

$P \wedge Q$ is logically equivalent to the sentence $Q \wedge P$, which is most naturally read as formalizing 'Alice got pregnant and married'. But in some circumstances, we intend to convey different things with these two sentences; in particular, they might be used to convey information about the temporal order of the relevant events. In this case, the sentence must be formalized simply as P, where

P: Alice married and got pregnant.

(iv) The sentence is ambiguous. It is most naturally read as saying that Anne and Barbara carried the piano together. On the less natural reading, it says that Anne and Barbara each carried the piano individually – presumably at different moments in time. The first reading can be formalized as $P \wedge Q \wedge R$, where

P: Anne and Barbara carried the piano.
Q: Anne sweated.
R: Barbara sweated.

The second reading can be formalized as $P_1 \wedge P_2 \wedge Q_1 \wedge Q_2$, where

P_1: Anne carried the piano.
P_1: Barbara carried the piano.
Q_1: Anne sweated.
Q_2: Barbara sweated.

Answer to 3.7. (i) Premises: P, $P \to Q$; conclusion: Q.

P: Antonia is a mammal.
Q: Antonia is mortal.

$P, P \to Q \vDash Q$, so the argument is propositionally valid, and therefore also valid.

(ii) Premises: P, Q; conclusion: R.

P: Antonia is a mammal.
Q: Every mammal is mortal.
R: Antonia is mortal.

R does not follow from P and Q, so the argument is not propositionally valid. It is valid.

(iii) Premises: P, $\neg P$; conclusion: Q.

P: Antonia is a mammal.
Q: Antonia is mortal.

$P, \neg P \vDash Q$, so the argument is propositionally valid, and therefore also valid.

The argument might strike one as invalid, but it is clearly valid according to Characterization 1.9; see the discussion in section 3.6 of the *Manual*.

(iv) It is natural to try to formalize the premises using $\neg(P \to Q)$ and $\neg P$ and the conclusion using Q, where

P: Antonia is a mammal.
Q: Antonia is mortal.

$\neg(P \to Q), \neg P \vDash Q$, so if this was an adequate formalization, it would follow that the argument is propositionally valid, and therefore also valid.

As in (iii), the argument might strike one as invalid. And as in (iii), one might try to explain this away by arguing that we are confused by the fact that the premises are inconsistent (note that P is true in any $\mathcal{L}_1$-structure in which $\neg(P \to Q)$ is true). But informally, the premises are plausibly *not* inconsistent. This is because it is natural to read the first premise ('It is not the case that if Antonia is a mammal, she is mortal.') as expressing only that it does not *follow* (in some sense) from Antonia being a mammal that she is mortal. Thus in this case, 'if … then …' might not be correctly formalized using $\to$. Therefore, it might be better to formalize the premises using $\neg P$ and $\neg Q$, and the conclusion using R, where

P: If Antonia is a mammal, she is mortal.
Q: Antonia is a mammal.
R: Antonia is mortal.

R does not follow from $\neg P$ and $\neg Q$, so the argument would then not be propositionally valid, and it would therefore no longer follow that it is valid.

Which of these formalizations is in fact the most adequate one and whether the argument is in fact valid depends on subtle issues concerning 'if … then …' in English, which we can't settle here.

4 The Syntax of Predicate Logic

Exercise 4.1. Determine which of the following expressions are formulas of $\mathcal{L}_2$ and which are also sentences of $\mathcal{L}_2$. For every formula of $\mathcal{L}_2$, add the omitted arity indices to all predicate letters and mark all free occurrences of variables.

(i) $\exists x(\forall x Rxx \land \neg\exists x Qxx)$
(ii) $\forall a(Pa \to Qy)$
(iii) $\exists y_2(Ry_1y_3)$
(iv) $\exists z_{44}(Pz_{44}z_{44} \lor Pz_{44}z_4z_4)$
(v) $\exists x(\exists x Rxy \leftrightarrow \exists y Rxy)$
(vi) $\forall z Pcz$
(vii) $\forall x P_1xyz \land \forall y P_2xyz \land \forall z P_3xyz$
(viii) $\neg(\exists x Rxa)$

Exercise 4.2. Formalize the following English sentences in $\mathcal{L}_2$. Make the formalizations as detailed as possible, and explicitly specify a dictionary.

(i) Anton is a zebra.
(ii) All marmots are furry.
(iii) Some whales are large mammals.
(iv) Some chameleons can run faster than every penguin.
(v) The octopus is using the coconut shell to hide from the moray.
(vi) Tina likes all llamas.
(vii) Tina likes every llama which likes itself.
(viii) Tina likes all llamas which like all llamas.
(ix) Tina likes all and only those llamas which like all and only those llamas Tina likes.

Exercise 4.3. Formalize the following English sentences in the language $\mathcal{L}_2$ of predicate logic using the dictionary below.

a:	the empty set
P^1:	… is a set
Q^1:	… is an ordered pair
R^1:	… is a binary relation
R^2:	… has … as an element

(i) Something is not a set.
(ii) Not every set is an ordered pair.
(iii) The empty set is both a set and a binary relation, but not an ordered pair.
(iv) The empty set has no elements.
(v) Everything is an element of some set, but no set has everything as an element.
(vi) Anything which has itself as an element is not a set.
(vii) A set is a binary relation if and only if it has only ordered pairs as elements.
(viii) For every set, there is a set whose elements are exactly the sets whose elements are elements of the first set.
(ix) Every non-empty set has an element which is either not a set or has no elements which are also elements of the first set.

Exercise 4.4. Translate the following sentences of $\mathcal{L}_2$ into idiomatic English using the following dictionary:

R^2: … is a part of …

(i) $\forall x Rxx$
(ii) $\forall x \exists y Rxy$
(iii) $\forall y \exists x Rxy$
(iv) $\exists x \forall y Rxy$
(v) $\exists y \forall x Rxy$
(vi) $\forall x \forall y \exists z(Rxz \wedge Ryz)$
(vii) $\forall x \exists y(Ryx \wedge \neg \exists z(Rzy \wedge \neg Ryz))$

Answer to 4.1. We mark free variables by underlining them.

(i) $\exists x(\forall x R^2 xx \wedge \neg\exists x Q^2 xx)$ is a formula and a sentence.
(ii) $\forall a(Pa \to Qy)$ is not a formula of $\mathcal{L}_2$ since a is not a variable.
(iii) $\exists y_2(Ry_1y_3)$ is not a formula of $\mathcal{L}_2$ because of the presence of the brackets.
(iv) $\exists z_{44}(P^2 z_{44}z_{44} \vee P^3 z_{44}\underline{z_4z_4})$ is a formula but not a sentence.
(v) $\exists x(\exists x R^2 x\underline{y} \leftrightarrow \exists y R^2 xy)$ is a formula but not a sentence.
(vi) $\forall z P^2 cz$ is a formula and a sentence.
(vii) $\forall x P^3_1 x\underline{y}\underline{z} \wedge \forall y P^3_2 \underline{x}y\underline{z} \wedge \forall z P^3_3 \underline{x}\underline{y}z$ is a formula but not a sentence.
(viii) $\neg(\exists x Rxa)$ is not a formula of $\mathcal{L}_2$ because of the presence of the brackets.

Answer to 4.2.

(i) Pa

a:	Anton
P^1:	… is a zebra

(ii) $\forall x(Px \to Qx)$

P^1:	… is a marmot
Q^1:	… is furry

(iii) $\exists x(Px \wedge Qx)$

P^1:	… is a whale
Q^1:	… is a large mammal

(iv) $\exists x(Px \wedge \forall y(Qy \to Rxy))$

P^1:	… is chameleon
Q^1:	… is a penguin
R^2:	… can run faster than …

(v) $Pabc$

a:	the octopus
b:	the coconut shell
c:	the moray
P^3:	… is using … to hide from ….

(vi) $\forall x(Px \to Qax)$
(vii) $\forall x(Px \land Qxx \to Qax)$
(viii) $\forall x(Px \land \forall y(Py \to Qxy) \to Qax)$
(ix) $\forall x(Px \to (Qax \leftrightarrow \forall y(Py \to (Qxy \leftrightarrow Qay))))$

a:	Tina
P^1:	… is a llama
Q^2:	… likes …

Answer to 4.3.

(i) $\exists x \neg Px$
(ii) $\neg \forall x(Px \to Qx)$
(iii) $Pa \land Ra \land \neg Qa$
(iv) $\neg \exists x Rax$
(v) $\forall x \exists y(Py \land Ryx) \land \neg \exists x(Px \land \forall y Rxy)$
(vi) $\forall x(Rxx \to \neg Px)$
(vii) $\forall x(Px \to (Rx \leftrightarrow \forall y(Rxy \to Qy)))$
(viii) $\forall x(Px \to \exists y(Py \land \forall z(Ryz \leftrightarrow (Pz \land \forall x_1(Rzx_1 \to Rxx_1)))))$
(ix) $\forall x(Px \land \exists y Rxy \to \exists y(Rxy \land (\neg Py \lor \neg \exists z(Ryz \land Rxz))))$

Answer to 4.4.

(i) Everything is a part of itself.
(ii) Everything is a part of something.
(iii) Everything has something as a part.
(iv) Something is a part of everything.
(v) Something has everything as a part.
(vi) For any two things, there is something of which they are both a part.
(vii) Everything has a part which has no part of which it is not also a part.

5 The Semantics of Predicate Logic

Exercise 5.1. Consider an $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1,2\} \\ |R|_{\mathcal{A}} &= \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\} \\ |a|_{\mathcal{A}} &= 0 \\ |b|_{\mathcal{A}} &= 1 \end{aligned}$$

Determine whether the following sentences are true or false in $\mathcal{A}$. Sketch proofs for your answers.

(i) $Rab \land (Rba \to \exists x Rxx)$
(ii) $\exists x \exists y (Rxy \land Ryx)$
(iii) $\forall x \exists y (Rxy \lor Ryx)$
(iv) $\forall x \forall y \forall z ((Rxy \land Ryz) \to Rxz)$
(v) $\forall x (Rxb \leftrightarrow \exists y (Ryb \land Rxy))$

Exercise 5.2. Refute each of the following claims by means of a counterexample. You only need to specify the counterexample; there is no need to show that your structure is a counterexample.

(i) $\neg \forall x (Px \to \neg Px)$ is logically true.
(ii) $\forall x \exists y Pxyy \lor \exists x \neg \exists y \exists z Pxyz$ is logically true.
(iii) $\forall x (Px \to \exists y (Ryx \land Py)) \vDash \neg \forall x Px$
(iv) $\forall y (\exists x Ryx \land \exists x Qyx) \vDash \forall y \exists x (Ryx \land Qyx)$
(v) $\forall x \exists y \exists z (Qxy \land Qzx) \vDash \forall x \forall y \forall z (Qxy \to (Qyz \to Qxz))$
(vi) $\forall x \forall y (Px \to (Qy \to \neg Rxyx)) \vDash \forall x_1 \forall z (Rx_1 z x_1 \to (Qz \land \neg Px_1))$

Exercise 5.3. Specify for each of the following sentences an $\mathcal{L}_2$-structure in which it is true.

(i) $\forall x(Px \to \exists y(Py \land Rxy))$
(ii) $(Rab \land \neg Rba) \land \forall x \forall y \forall z(Rxy \to (Ryz \to Rxz)) \land \forall x Rxx$
(iii) $\forall x \exists y \exists z_1(Ryxz_1 \land Pz_1 \land \exists z_2(Ryxz_2 \land \neg Pz_2))$

Exercise 5.4. Recall that a set of $\mathcal{L}_2$-sentences is semantically consistent if there is an $\mathcal{L}_2$-structure in which all members of the set are true. Specify structures to show that each of the following sets is semantically consistent.

(i) $\{\neg Pc, ((Pa \land Pb) \to Pc), Pa\}$
(ii) $\{\forall x \forall y(Rxy \leftrightarrow \neg Qyx), \forall x \neg Rxx\}$
(iii) $\{\forall x(Rxx \lor Qxx), \forall x \forall y(Rxy \leftrightarrow \neg Qxy), \neg \forall x Rxx, \neg \forall x Qxx\}$
(iv) $\{\forall x \forall y \forall z(\neg Rxy \lor \neg Ryz \lor Rxz), \forall x \forall y(\neg Rxy \lor Ryx), \neg \forall x Rxx\}$

Exercise 5.5. Consider the following $\mathcal{L}_2$-sentence:

$$\forall x \exists y \exists z(Qxyz \land Py \land \neg Pz \land \neg Qxxz \land \neg Qxyx)$$

Specify an $\mathcal{L}_2$-structure with as few elements in its domain as possible in which the sentence is true. Sketch an argument why the sentence is true in the structure you have specified and explain why it is not true in any $\mathcal{L}_2$-structure with fewer elements in its domain.

Exercise 5.6. Note that the syntax of $\mathcal{L}_2$ contains the sentence letters of $\mathcal{L}_1$ as 0-place predicate letters and that the syntax of $\mathcal{L}_2$ allows all of the connectives of $\mathcal{L}_1$ in building formulas. Consequently, every sentence of $\mathcal{L}_1$ is a sentence of $\mathcal{L}_2$.

For any set Γ of $\mathcal{L}_1$-sentences and $\mathcal{L}_1$-sentence ϕ, show that Definitions 2.9 and 5.8 agree on whether ϕ follows from Γ. I.e., show that $\Gamma \vDash \phi$ according to Definition 2.9 if and only if $\Gamma \vDash \phi$ according to Definition 5.8.

Exercise 5.7. Specify a semantically consistent set X of $\mathcal{L}_2$-sentences containing no other than unary predicate letters such that every $\mathcal{L}_2$-structure in which all of its elements are true has an infinite domain.

Show that for every finite set Y of elements of X, there is an $\mathcal{L}_2$-structure with a finite domain in which all elements of Y are true.

Exercise 5.8. Recall that $\mathcal{L}_2$-structures assign T or F to 0-ary predicate letters (i.e., sentence letters), and that for every natural number $n \geq 1$, they assign n-ary relations to n-ary predicate letters. Recall also the satisfaction clause for atomic $\mathcal{L}_2$-sentences:

(i) $|\Phi t_1 \dots t_n|^\alpha_\mathcal{A} = \mathrm{T}$ if and only if $\langle |t_1|^\alpha_\mathcal{A}, \dots, |t_n|^\alpha_\mathcal{A} \rangle \in |\Phi|^\alpha_\mathcal{A}$, where Φ is an n-ary predicate letter (n must be 1 or higher), and each of $t_1, \dots, t_n$ is either a variable or a constant.

There are unique sets X and Y such that if $\mathrm{T} = X$ and $\mathrm{F} = Y$, then we can drop the special treatment of sentence letters in both the definition of $\mathcal{L}_2$-structures and the satisfaction clauses. Which sets are X and Y?

Answer to 5.1.

(i) $Rab \wedge (Rba \to \exists x Rxx)$ is true in $\mathcal{A}$. We first show that Rab is true in $\mathcal{A}$:

$$\langle 0, 1\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$$
$$\langle |a|_{\mathcal{A}}, |b|_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$$
$$|Rab|_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(i)} \qquad (*)$$

We now show that Rba is false in $\mathcal{A}$:

$$\langle 1, 0\rangle \notin \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$$
$$\langle |b|_{\mathcal{A}}, |a|_{\mathcal{A}}\rangle \notin |R|_{\mathcal{A}}$$
$$|Rba|_{\mathcal{A}} = \mathrm{F} \qquad \text{Definition 5.2(i)}$$
$$|Rba \to \exists x Rxx|_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(v)}$$
$$|Rab \wedge (Rba \to \exists x Rxx)|_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(iii) and line } (*)$$

(ii) $\exists x \exists y(Rxy \wedge Ryx)$ is true in $\mathcal{A}$. Let α be a variable assignment over $\mathcal{A}$ such that $|x|^{\alpha}_{\mathcal{A}} = 0$ and $|y|^{\alpha}_{\mathcal{A}} = 2$. Then:

$$\langle 0, 2\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$$
$$\langle |x|^{\alpha}_{\mathcal{A}}, |y|^{\alpha}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$$
$$|Rxy|^{\alpha}_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(i)} \qquad (*)$$
$$\langle 2, 0\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$$
$$\langle |y|^{\alpha}_{\mathcal{A}}, |x|^{\alpha}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$$
$$|Ryx|^{\alpha}_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(i)}$$
$$|Rxy \wedge Ryx|^{\alpha}_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(iii) and line } (*)$$
$$|\exists y(Rxy \wedge Ryx)|^{\alpha}_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(viii)}$$
$$|\exists x \exists y(Rxy \wedge Ryx)|_{\mathcal{A}} = \mathrm{T} \qquad \text{Definition 5.2(viii)}$$

(iii) $\forall x \exists y(Rxy \vee Ryx)$ is true in $\mathcal{A}$. Let α be any variable assignment over $\mathcal{A}$. We distinguish three cases:
First case. $|x|^{\alpha}_{\mathcal{A}} = 0$. Let β be a variable assignment over $\mathcal{A}$ differing from α in y at most such that $|y|^{\beta}_{\mathcal{A}} = 1$. Then:

$\langle 0, 1\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$	
$\langle |x|^{\beta}_{\mathcal{A}}, |y|^{\beta}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$	
$|Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(i)
$|Rxy \vee Ryx|^{\beta}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(iv)
$|\exists y(Rxy \vee Ryx)|^{\alpha}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(viii)

Second case. $|x|^{\alpha}_{\mathcal{A}} = 1$. Let β be a variable assignment over $\mathcal{A}$ differing from α in y at most such that $|y|^{\beta}_{\mathcal{A}} = 0$. Then:

$\langle 0, 1\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$	
$\langle |y|^{\beta}_{\mathcal{A}}, |x|^{\beta}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$	
$|Ryx|^{\beta}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(i)
$|Rxy \vee Ryx|^{\beta}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(iv)
$|\exists y(Rxy \vee Ryx)|^{\alpha}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(viii)

Third case. $|x|^{\alpha}_{\mathcal{A}} = 2$. Let β be a variable assignment over $\mathcal{A}$ differing from α in y at most such that $|y|^{\beta}_{\mathcal{A}} = 0$. Then:

$\langle 2, 0\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$	
$\langle |x|^{\beta}_{\mathcal{A}}, |y|^{\beta}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$	
$|Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(i)
$|Rxy \vee Ryx|^{\beta}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(iv)
$|\exists y(Rxy \vee Ryx)|^{\alpha}_{\mathcal{A}} = \mathrm{T}$	Definition 5.2(viii)

That $|\forall x \exists y(Rxy \vee Ryx)|_{\mathcal{A}} = \mathrm{T}$ follows from these three cases by Definition 5.2(vii).

(iv) $\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$ is false in $\mathcal{A}$. Let α be any variable assignment over $\mathcal{A}$ such that $|x|^{\alpha}_{\mathcal{A}} = |z|^{\alpha}_{\mathcal{A}} = 0$ and $|y|^{\alpha}_{\mathcal{A}} = 2$. Then:

$\langle 0,2\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$		
$\langle |x|^{\alpha}_{\mathcal{A}}, |y|^{\alpha}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$		
$|Rxy|^{\alpha}_{\mathcal{A}} = \text{T}$	Definition 5.2(i)	(*)
$\langle 2,0\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$		
$\langle |y|^{\alpha}_{\mathcal{A}}, |z|^{\alpha}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$		
$|Ryz|^{\alpha}_{\mathcal{A}} = \text{T}$	Definition 5.2(i)	
$|Rxy \wedge Ryz|^{\alpha}_{\mathcal{A}} = \text{T}$	Definition 5.2(iii) and line (*)	(†)
$\langle 0,0\rangle \notin \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$		
$\langle |x|^{\alpha}_{\mathcal{A}}, |z|^{\alpha}_{\mathcal{A}}\rangle \notin |R|_{\mathcal{A}}$		
$|Rxz|^{\alpha}_{\mathcal{A}} = \text{F}$	Definition 5.2(i)	
$|(Rxy \wedge Ryz) \to Rxz|^{\alpha}_{\mathcal{A}} = \text{F}$	Definition 5.2(v) and line (†)	
$|\forall z(Rxy \wedge Ryz) \to Rxz|^{\alpha}_{\mathcal{A}} = \text{F}$	Definition 5.2(vii)	
$|\forall y \forall z(Rxy \wedge Ryz) \to Rxz|^{\alpha}_{\mathcal{A}} = \text{F}$	Definition 5.2(vii)	
$|\forall x \forall y \forall z(Rxy \wedge Ryz) \to Rxz|_{\mathcal{A}} = \text{F}$	Definition 5.2(vii)	

(v) $\forall x(Rxb \leftrightarrow \exists y(Ryb \wedge Rxy))$ is true in $\mathcal{A}$. Let α be any variable assignment over $\mathcal{A}$. We distinguish three cases.
First case. $|x|^{\alpha}_{\mathcal{A}} = 0$. Then:

$\langle 0,1\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$		
$\langle |x|^{\alpha}_{\mathcal{A}}, |b|_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}}$		
$|Rxb|^{\alpha}_{\mathcal{A}} = \text{T}$	Definition 5.2(i)	(1)

Let β be a variable assignment over $\mathcal{A}$ differing from α in y at most such that $|y|^{\beta}_{\mathcal{A}} = 2$. Then:

$$\begin{array}{lll}
\langle 2,1\rangle \in \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & & \\
\langle |y|^{\beta}_{\mathcal{A}}, |b|_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}} & & \\
|Ryb|^{\beta}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(i)} & (2) \\
\langle 0,2\rangle \in \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & & \\
\langle |x|^{\beta}_{\mathcal{A}}, |y|^{\beta}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}} & & \\
|Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(i)} & \\
|Ryb \wedge Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(iii) and line (2)} & \\
|\exists y(Ryb \wedge Rxy)|^{\alpha}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(viii)} & \\
|Rxb \leftrightarrow \exists y(Ryb \wedge Rxy)|^{\alpha}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(vi) and line (1)} & (3)
\end{array}$$

Second case. $|x|^{\alpha}_{\mathcal{A}} = 1$. Then:

$$\begin{array}{lll}
\langle 1,1\rangle \notin \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & & \\
\langle |x|^{\alpha}_{\mathcal{A}}, |b|_{\mathcal{A}}\rangle \notin |R|_{\mathcal{A}} & & \\
|Rxb|^{\alpha}_{\mathcal{A}} = \mathrm{F} & \text{Definition 5.2(i)} & (4)
\end{array}$$

Let β be a variable assignment over $\mathcal{A}$ differing from α in y at most. Then:

$$\begin{array}{ll}
\langle 1, |y|^{\beta}_{\mathcal{A}}\rangle \notin \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & \\
\langle |x|^{\beta}_{\mathcal{A}}, |y|^{\beta}_{\mathcal{A}}\rangle \notin |R|_{\mathcal{A}} & \\
|Rxy|^{\beta}_{\mathcal{A}} = \mathrm{F} & \text{Definition 5.2(i)} \\
|Ryb \wedge Rxy|^{\beta}_{\mathcal{A}} = \mathrm{F} & \text{Definition 5.2(iii)} \\
|\exists y(Ryb \wedge Rxy)|^{\alpha}_{\mathcal{A}} = \mathrm{F} & \text{Definition 5.2(viii)} \\
|Rxb \leftrightarrow \exists y(Ryb \wedge Rxy)|^{\alpha}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(vi) and line (4)}
\end{array}$$

Third case. $|x|^{\alpha}_{\mathcal{A}} = 2$. Then:

$$\begin{array}{lll}
\langle 2,1\rangle \in \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & & \\
\langle |x|^{\alpha}_{\mathcal{A}}, |b|_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}} & & \\
|Rxb|^{\alpha}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(i)} & (5)
\end{array}$$

Let β be a variable assignment over $\mathcal{A}$ differing from α in y at most such that $|y|^{\beta}_{\mathcal{A}} = 0$. Then:

$$\begin{array}{lll}
\langle 0,1\rangle \in \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & & \\
\langle |y|^{\beta}_{\mathcal{A}}, |b|_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}} & & \\
|Ryb|^{\beta}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(i)} & (6) \\
\langle 2,0\rangle \in \{\langle 0,1\rangle,\langle 0,2\rangle,\langle 2,1\rangle,\langle 2,0\rangle\} & & \\
\langle |x|^{\beta}_{\mathcal{A}}, |y|^{\beta}_{\mathcal{A}}\rangle \in |R|_{\mathcal{A}} & & \\
|Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(i)} & \\
|Ryb \wedge Rxy|^{\beta}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(iii) and line (6)} & \\
|\exists y(Ryb \wedge Rxy)|^{\alpha}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(viii)} & \\
|Rxb \leftrightarrow \exists y(Ryb \wedge Rxy)|^{\alpha}_{\mathcal{A}} = \mathrm{T} & \text{Definition 5.2(vi) and line (5)} &
\end{array}$$

That $|\forall x(Rxb \leftrightarrow \exists y(Ryb \wedge Rxy))|_{\mathcal{A}} = \mathrm{T}$ follows from these three cases by Definition 5.2(vii).

Answer to 5.2.

(i) $\neg\forall x(Px \to \neg Px)$ is false in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that $|P|_{\mathcal{A}} = \varnothing$.

(ii) $\forall x\exists yPxyy \vee \exists x\neg\exists y\exists zPxyz$ is false in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$D_{\mathcal{A}} = \{0,1\}$$
$$|P|_{\mathcal{A}} = \{\langle 0,0,1\rangle, \langle 1,0,1\rangle\}$$

(iii) A counterexamples to $\forall x(Px \to \exists y(Ryx \wedge Py)) \vDash \neg\forall xPx$ is any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$D_{\mathcal{A}} = \{0\}$$
$$|P|_{\mathcal{A}} = \{0\}$$
$$|R|_{\mathcal{A}} = \{\langle 0,0\rangle\}$$

(iv) A counterexamples to $\forall y(\exists x Ryx \wedge \exists x Qyx) \vDash \forall y \exists x(Ryx \wedge Qyx)$ is any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |Q|_{\mathcal{A}} &= \{\langle 0,1\rangle, \langle 1,0\rangle\} \\ |R|_{\mathcal{A}} &= \{\langle 0,0\rangle, \langle 1,1\rangle\} \end{aligned}$$

(v) A counterexamples to $\forall x \exists y \exists z(Qxy \wedge Qzx) \vDash \forall x \forall y \forall z(Qxy \rightarrow (Qyz \rightarrow Qxz))$ is any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1,2\} \\ |Q|_{\mathcal{A}} &= \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\} \end{aligned}$$

(vi) A counterexamples to $\forall x \forall y(Px \rightarrow (Qy \rightarrow \neg Rxyx)) \vDash \forall x_1 \forall z(Rx_1zx_1 \rightarrow (Qz \wedge \neg Px_1))$ is any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |P|_{\mathcal{A}} &= \varnothing \\ |Q|_{\mathcal{A}} &= \varnothing \\ |R|_{\mathcal{A}} &= \{\langle 0,1,0\rangle\} \end{aligned}$$

Answer to 5.3.

(i) The sentence $\forall x(Px \rightarrow \exists y(Py \wedge Rxy))$ is true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that $|P|_{\mathcal{A}} = \varnothing$.

(ii) $(Rab \wedge \neg Rba) \wedge \forall x \forall y \forall z(Rxy \rightarrow (Ryz \rightarrow Rxz)) \wedge \forall x Rxx$ is true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |R|_{\mathcal{A}} &= \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\} \\ |a|_{\mathcal{A}} &= 0 \\ |b|_{\mathcal{A}} &= 1 \end{aligned}$$

(iii) $\forall x \exists y \exists z_1 (Ryxz_1 \land Pz_1 \land \exists z_2 (Ryxz_2 \land \neg Pz_2))$ is true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |P|_{\mathcal{A}} &= \{0\} \\ |R|_{\mathcal{A}} &= \{\langle 0,0,0\rangle, \langle 0,0,1\rangle, \langle 0,1,0\rangle, \langle 0,1,1\rangle\} \end{aligned}$$

Answer to 5.4.

(i) All members of $\{\neg Pc, ((Pa \land Pb) \to Pc), Pa\}$ are true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |P|_{\mathcal{A}} &= \{0\} \\ |a|_{\mathcal{A}} &= 0 \\ |b|_{\mathcal{A}} &= 1 \\ |c|_{\mathcal{A}} &= 1 \end{aligned}$$

(ii) All members of $\{\forall x \forall y (Rxy \leftrightarrow \neg Qyx), \forall x \neg Rxx\}$ are true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0\} \\ |Q|_{\mathcal{A}} &= \{\langle 0,0\rangle\} \\ |R|_{\mathcal{A}} &= \varnothing \end{aligned}$$

(iii) All members of $\{\forall x (Rxx \lor Qxx), \forall x \forall y (Rxy \leftrightarrow \neg Qxy), \neg \forall x Rxx, \neg \forall x Qxx\}$ are true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |Q|_{\mathcal{A}} &= \{\langle 0,0\rangle, \langle 0,1\rangle\} \\ |R|_{\mathcal{A}} &= \{\langle 1,0\rangle, \langle 1,1\rangle\} \end{aligned}$$

(iv) All members of $\{\forall x \forall y \forall z (\neg Rxy \lor \neg Ryz \lor Rxz), \forall x \forall y (\neg Rxy \lor Ryx), \neg \forall x Rxx\}$ are true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that $|R|_{\mathcal{A}} = \varnothing$.

Answer to 5.5. $\forall x \exists y \exists z(Qxyz \wedge Py \wedge \neg Pz \wedge \neg Qxxz \wedge \neg Qxyx)$ is true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1,2,3\} \\ |P|_{\mathcal{A}} &= \{0,1\} \\ |Q|_{\mathcal{A}} &= \{\langle 0,1,2\rangle, \langle 1,0,2\rangle, \langle 2,0,3\rangle, \langle 3,0,2\rangle\} \end{aligned}$$

Consider any $\mathcal{L}_2$-structure $\mathcal{B}$ in which $\forall x \exists y \exists z(Qxyz \wedge Py \wedge \neg Pz \wedge \neg Qxxz \wedge \neg Qxyx)$ is true. Let α be an assignment over $\mathcal{B}$. Then there is an assignment β over $\mathcal{B}$ differing from α in at most y and z such that $|Qxyz \wedge Py \wedge \neg Pz \wedge \neg Qxxz \wedge \neg Qxyx|^{\beta}_{\mathcal{B}} = \mathrm{T}$. From the truth of Py and $\neg Pz$, it follows that $|y|^{\beta}_{\mathcal{B}} \neq |z|^{\beta}_{\mathcal{B}}$. Similarly, by the truth of $Qxyz$ and $\neg Qxxz$, it follows that $|x|^{\beta}_{\mathcal{B}} \neq |y|^{\beta}_{\mathcal{B}}$, and by the truth of $Qxyz$ and $\neg Qxyx$, it follows that $|x|^{\beta}_{\mathcal{B}} \neq |z|^{\beta}_{\mathcal{B}}$. Thus $D_{\mathcal{B}}$ has at least three elements. Now, either $|Px|^{\alpha}_{\mathcal{B}} = \mathrm{T}$ or $|Px|^{\alpha}_{\mathcal{B}} = \mathrm{F}$. In the former case, consider an assignment γ over $\mathcal{B}$ such that $|x|^{\gamma}_{\mathcal{B}} = |z|^{\beta}_{\mathcal{B}}$; in the latter case, consider an assignment γ over $\mathcal{B}$ such that $|x|^{\gamma}_{\mathcal{B}} = |y|^{\beta}_{\mathcal{B}}$. In either case, as before, it follows that there is an assignment δ over $\mathcal{B}$ differing from γ at most in y and z such that all of $|x|^{\delta}_{\mathcal{B}}$, $|y|^{\delta}_{\mathcal{B}}$ and $|z|^{\delta}_{\mathcal{B}}$ are distinct, $|y|^{\delta}_{\mathcal{B}} \in |P|_{\mathcal{B}}$ and $|z|^{\delta}_{\mathcal{B}} \notin |P|_{\mathcal{B}}$. This is impossible if $D_{\mathcal{B}} = \{|x|^{\beta}_{\mathcal{B}}, |y|^{\beta}_{\mathcal{B}}, |z|^{\beta}_{\mathcal{B}}\}$, hence $D_{\mathcal{B}}$ has at least four members.

Answer to 5.6. Assume $\Gamma \nvDash \phi$ according to Definition 2.9. Then there is an $\mathcal{L}_1$-structure $\mathcal{A}$ such that all members of Γ are true in $\mathcal{A}$ and ϕ is false in $\mathcal{A}$. Extend $\mathcal{A}$ to an $\mathcal{L}_2$-structure $\mathcal{A}'$ by adding a domain and interpretations of constants and n-ary predicate letters for $n \geq 1$. Since the satisfaction clauses of sentence letters and the propositional connectives of $\mathcal{L}_1$ are the same in $\mathcal{L}_1$ and $\mathcal{L}_2$, all members of Γ are true in $\mathcal{A}'$ and ϕ is false in $\mathcal{A}'$. So according to Definition 5.8, $\Gamma \nvDash \phi$.

Assume $\Gamma \nvDash \phi$ according to Definition 5.8. Then there is an $\mathcal{L}_2$-structure $\mathcal{A}$ such that all members of Γ are true in $\mathcal{A}$ and ϕ is false in $\mathcal{A}$. Derive an $\mathcal{L}_1$-structure $\mathcal{A}'$ from $\mathcal{A}$ by omitting the domain and the interpretations of constants and n-ary predicate letters for $n \geq 1$. Since the satisfaction clauses of sentence letters and the propositional connectives of $\mathcal{L}_1$ are the same in $\mathcal{L}_1$ and $\mathcal{L}_2$, all members of Γ are true in $\mathcal{A}'$ and ϕ is false in $\mathcal{A}'$. So according to Definition 2.9, $\Gamma \nvDash \phi$.

Answer to 5.7. I will write $\mathbb{N}$ for the set of natural numbers. Let X be the set of sentences which are members of one of the following sets: $\{P_i a_j : i, j \in \mathbb{N}$ and $i \leq j\}$ and $\{\neg P_i a_j : i, j \in \mathbb{N}$ and $i > j\}$. X is semantically consistent, since all of its elements are true in any $\mathcal{L}_2$-structure $\mathcal{A}$ such that $D_\mathcal{A} = \mathbb{N}$, and for every $i \in \mathbb{N}$:

$$|P_i|_\mathcal{A} = \{j : j \in \mathbb{N} \text{ and } i \leq j\}$$
$$|a_i|_\mathcal{A} = i$$

Consider any finite set Y of elements of X. If $Y = \varnothing$, it is trivially true that all of its elements are true in every $\mathcal{L}_2$-structure with a finite domain. If $Y \neq \varnothing$, let i be the largest natural number such that there is a natural number j such that one of $P_i a_j$, $\neg P_i a_j$, $P_j a_i$ and $\neg P_j a_i$ is a member of Y. Let $\mathcal{B}$ be an $\mathcal{L}_2$-structure such that $D_\mathcal{B} = \{j : j \in \mathbb{N}$ and $j \leq i\}$, and for every $j \in \mathbb{N}$ such that $j \leq i$:

$$|P_j|_\mathcal{B} = \{k : k \in \mathbb{N} \text{ and } j \leq k\}$$
$$|a_j|_\mathcal{B} = j$$

All elements of Y are true in $\mathcal{B}$.

Answer to 5.8. For us to be able to treat sentence letters like other predicate letters, sentence letters have to be interpreted by $\mathcal{L}_2$-structures as 0-ary relations. Since a 0-ary relation is a set of 0-tuples, and there is only one 0-tuple, namely $\langle\rangle$, there are exactly two 0-ary relations: $\varnothing$ and $\{\langle\rangle\}$.

Applying the satisfaction clause of atomic sentences to sentence letters, we obtain the following condition:

(i_0) $|\Phi|_\mathcal{A} = \mathrm{T}$ if and only if $\langle\rangle \in |\Phi|^\alpha_\mathcal{A}$, where Φ is a 0-ary predicate letter.

So T must be $\{\langle\rangle\}$ and F must be $\varnothing$.

6 Natural Deduction

Exercise 6.1. Establish the following claims by means of proofs in the system of Natural Deduction.

(i) $P \to (Q \leftrightarrow R) \vdash P \wedge Q \leftrightarrow P \wedge R$
(ii) $\neg((P \to \neg Q) \vee \neg R) \vdash Q$
(iii) $P \vee \neg Q \to R \vdash \neg R \to \neg P \wedge Q$

Exercise 6.2. Establish the following claims by means of proofs in the system of Natural Deduction.

(i) $\exists x(Px \wedge \forall y Rxy) \vdash \exists z Rzz$
(ii) $\forall x \exists y Rxy \vdash \exists y \neg \forall z(Ray \to \neg Ryz)$
(iii) $\forall y \exists x(Ryx \vee Qyx) \vdash \forall y(\exists x Ryx \vee \exists x Qyx)$
(iv) $(\exists x Px \wedge \exists y Qy) \vee (\neg \exists x Px \wedge \neg \exists y Qy) \vdash \exists x Px \leftrightarrow \exists y Qy$
(v) $\forall x \forall y(Px \to (Qy \to \neg Rxyx)) \vdash \forall x_1 \forall z(Rx_1 z x_1 \to (Qz \to \neg Px_1))$
(vi) $\forall x(Px \to \exists y(Ryx \wedge Py)), \forall x \forall y \forall z(Ryx \to (Rzy \to \neg Pz)) \vdash \neg \forall x Px$

Exercise 6.3. Consider the following attempted proofs. Briefly explain why they are incorrect, listing all mistakes in them. Where possible supply a corrected proof. Otherwise, provide a counterexample and briefly sketch the reasons why it is a counterexample.

(i) $\vdash Q \vee \neg Q$:

$$\dfrac{\dfrac{[Q]}{Q \vee \neg Q} \qquad \dfrac{[\neg Q]}{Q \vee \neg Q}}{Q \vee \neg Q}$$

(ii) $Pc \vdash \forall z(Qz \to Pz)$:

$$\dfrac{\dfrac{Pc}{Qc \to Pc}}{\forall z(Qz \to Pz)}$$

(iii) $\forall x\exists yRxy \vdash \exists x\exists y\exists z(Rxy \land Ryz)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\exists yRxy}{\exists yRby} \qquad \dfrac{\dfrac{\forall x\exists yRxy}{\exists yRay} \quad \dfrac{[Rab] \qquad [Rbc]}{Rab \land Rbc}}{Rab \land Rbc}}{Rab \land Rbc}}{\exists y(Ray \land Ryz)}}{\exists y\exists z(Ray \land Ryz)}}{\exists x\exists y\exists z(Rxy \land Ryz)}$$

Exercise 6.4. The notion of soundness is formally defined in section 7.1 of the *Logic Manual*. Consider the following variant of the Natural Deduction rule of universal introduction ($\forall$Intro):

> Assume that ϕ is a formula with at most v occurring freely and that ϕ does not contain the constant t. Assume further that there is a proof of $\phi[t/v]$. Then the result of appending $\forall v\, \phi$ to that proof is a proof of $\forall v\, \phi$.

(a) How does this modified rule differ from $\forall$Intro?
(b) Explain why replacing $\forall$Intro with this rule renders the system of Natural Deduction unsound.

Consider the rule described by the following graphical representation.

$$\dfrac{\begin{array}{c}[\phi]\\ \vdots\\ \psi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots\\ \chi\end{array}}{\phi \to \chi}$$

(c) State the rule in words.
(d) Would the system of Natural Deduction still be sound if this rule were added?

Answer to 6.1. (i) $P \to (Q \leftrightarrow R) \vdash P \land Q \leftrightarrow P \land R$:

$$\dfrac{\dfrac{\dfrac{[P \land Q]}{P} \quad \dfrac{\dfrac{\dfrac{[P \land Q]}{P} \quad P \to (Q \leftrightarrow R)}{Q \leftrightarrow R} \quad \dfrac{[P \land Q]}{Q}}{R}}{P \land R} \qquad \dfrac{\dfrac{[P \land R]}{P} \quad \dfrac{\dfrac{\dfrac{[P \land R]}{P} \quad P \to (Q \leftrightarrow R)}{Q \leftrightarrow R} \quad \dfrac{[P \land R]}{R}}{Q}}{P \land Q}}{P \land Q \leftrightarrow P \land R}$$

(ii) $\neg((P \to \neg Q) \lor \neg R) \vdash Q$:

$$\dfrac{\dfrac{\dfrac{[\neg Q]}{P \to \neg Q}}{(P \to \neg Q) \lor \neg R} \qquad \neg((P \to \neg Q) \lor \neg R)}{Q}$$

(iii) $P \lor \neg Q \to R \vdash \neg R \to \neg P \land Q$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P]}{P \lor \neg Q} \quad P \lor \neg Q \to R}{R} \qquad [\neg R]}{\neg P} \qquad \dfrac{\dfrac{\dfrac{[\neg Q]}{P \lor \neg Q} \quad P \lor \neg Q \to R}{R} \qquad [\neg R]}{Q}}{\neg P \land Q}}{\neg R \to \neg P \land Q}$$

Answer to 6.2. (i) $\exists x(Px \land \forall y Rxy) \vdash \exists z Rzz$:

$$\dfrac{\exists x(Px \land \forall y Rxy) \qquad \dfrac{\dfrac{\dfrac{[Pa \land \forall y Ray]}{\forall y Ray}}{Raa}}{\exists z Rzz}}{\exists z Rzz}$$

(ii) $\forall x\exists yRxy \vdash \exists y\neg\forall z(Ray \to \neg Ryz)$:

$$\dfrac{\dfrac{\forall x\exists yRxy}{\exists yRay} \quad \dfrac{\dfrac{\dfrac{\forall x\exists yRxy}{\exists yRby} \quad \dfrac{\dfrac{[Rab] \quad \dfrac{[\forall z(Rab \to \neg Rbz)]}{Rab \to \neg Rbc}}{\neg Rbc} \quad [Rbc]}{\neg\forall z(Rab \to \neg Rbz)}}{\neg\forall z(Rab \to \neg Rbz)}}{\exists y\neg\forall z(Ray \to \neg Ryz)}}{\exists y\neg\forall z(Ray \to \neg Ryz)}$$

(iii) $\forall y\exists x(Ryx \lor Qyx) \vdash \forall y(\exists xRyx \lor \exists xQyx)$:

$$\dfrac{\dfrac{\dfrac{\forall y\exists x(Ryx \lor Qyx)}{\exists x(Rax \lor Qax)} \quad \dfrac{[Rab \lor Qab] \quad \dfrac{\dfrac{[Rab]}{\exists xRax}}{\exists xRax \lor \exists xQax} \quad \dfrac{\dfrac{[Qab]}{\exists xQax}}{\exists xRax \lor \exists xQax}}{\exists xRax \lor \exists xQax}}{\exists xRax \lor \exists xQax}}{\forall y(\exists xRyx \lor \exists xQyx)}$$

(iv) For reasons of space, the proof of $(\exists xPx \land \exists yQy) \lor (\neg\exists xPx \land \neg\exists yQy) \vdash \exists xPx \leftrightarrow \exists yQy$ is given on page 50.
(v) $\forall x\forall y(Px \to (Qy \to \neg Rxyx)) \vdash \forall x_1\forall z(Rx_1zx_1 \to (Qz \to \neg Px_1))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Raba] \quad \dfrac{[Qb] \quad \dfrac{[Pa] \quad \dfrac{\dfrac{\forall x\forall y(Px \to (Qy \to \neg Rxyx))}{\forall y(Pa \to (Qy \to \neg Raya))}}{Pa \to (Qb \to \neg Raba)}}{Qb \to \neg Raba}}{\neg Raba}}{\neg Pa}}{Qb \to \neg Pa}}{Raba \to (Qb \to \neg Pa)}}{\forall z(Raza \to (Qz \to \neg Pa))}}{\forall x_1\forall z(Rx_1zx_1 \to (Qz \to \neg Px_1))}$$

(vi) For reasons of space, the proof of $\forall x(Px \to \exists y(Ryx \land Py)), \forall x\forall y\forall z(Ryx \to (Rzy \to \neg Pz)) \vdash \neg\forall xPx$ is given on page 50.

Answer to 6.3.

(i) The last step of the attempted proof does not conform to any of the rules; there is no rule which allows us to derive $Q \vee \neg Q$ from a proof of $Q \vee \neg Q$ with assumption Q and a proof of $Q \vee \neg Q$ with assumption $\neg Q$. In particular, note that it is not a correct application of disjunction elimination ($\vee$Elim), since this would require $Q \vee \neg Q$ as a third premise, which is of course exactly what we are trying to show. A correct proof is given as Example 6.6 in the *Logic Manual*.

(ii) The last step of the attempted proof does not conform to any of the rules. In particular, the rule of universal introduction ($\forall$Intro) is not applicable, as c occurs in the undischarged assumption Pc.
The following $\mathcal{L}_2$-structure $\mathcal{A}$ is a counterexample:

$$\begin{aligned} D_{\mathcal{A}} &= \{0, 1\} \\ |P|_{\mathcal{A}} &= \{0\} \\ |Q|_{\mathcal{A}} &= \{1\} \\ |c|_{\mathcal{A}} &= 0 \end{aligned}$$

(iii) The rule of existential elimination ($\exists$Elim) is used incorrectly twice. In the first instance, the introduced constant b occurs both in an undischarged assumption (Rbc) and the sentence which is intended to be derived in this step ($Rab \wedge Rbc$); in the second instance, the introduced constant c occurs in the sentence which is intended to be derived in this step ($Rab \wedge Rbc$).
The rule of existential introduction ($\exists$Intro) is also used incorrectly twice; we cannot derive $\exists y(Ray \wedge Ryz)$ from $Rab \wedge Rbc$ (note the replacement of c by z), nor can we derive $\exists y \exists z(Ray \wedge Ryz)$ from $\exists y(Ray \wedge Ryz)$.
The following is a correct proof:

$$\dfrac{\dfrac{\forall x \exists y Rxy}{\exists y Ray} \qquad \dfrac{\dfrac{\forall x \exists y Rxy}{\exists y Rby} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Rab] \qquad [Rbc]}{Rab \wedge Rbc}}{\exists z(Rab \wedge Rbz)}}{\exists y \exists z(Ray \wedge Ryz)}}{\exists x \exists y \exists z(Rxy \wedge Ryz)}}{\exists x \exists y \exists z(Rxy \wedge Ryz)}}{\exists x \exists y \exists z(Rxy \wedge Ryz)}$$

Answer to 6.4.

(a) This variant differs from $\forall$Intro in two ways: it omits the restriction that the proof of $\phi[t/v]$ may not contain any undischarged assumptions containing t, but adds the restriction that v may be the only variable occurring freely in ϕ.

(b) The proposed variant licenses the following derivation:

$$\frac{Pa}{\forall x Px}$$

However, $Pa \vDash \forall x Px$ is not the case, as can easily be shown using a counterexample. Thus in the proposed system, there a sentence provable from premises which does not follow from the premises (i.e., the argument constituted by the premises and the conclusion is not valid), which shows that the system is not sound.

(c) The rule can be read as follows: "The result of appending $\phi \to \chi$ to a proof of ψ and a proof of χ, discharging all assumptions of ϕ in the proof of ψ and discharging all assumptions of ψ in the proof of χ is a proof of $\phi \to \chi$."

The resulting system would still be sound. To see this, note that we can transform any proof using the new rule into a proof in the original system, by applying the following procedure to every application of the new rule: We replace every assumption of ψ in the proof of χ by a copy of the proof of ψ; this gives us a proof of χ, from which we can derive $\phi \to \chi$ using $\to$Intro, discharging all assumptions of ϕ.

6.2(iv)

$$\dfrac{(\exists xPx \land \exists yQy) \lor (\neg\exists xPx \land \neg\exists yQy) \quad \dfrac{\dfrac{[\exists xPx \land \exists yQy]}{\exists xPx} \quad \dfrac{[\exists xPx \land \exists yQy]}{\exists yQy}}{\exists xPx \leftrightarrow \exists yQy} \quad \dfrac{\dfrac{[\exists xPx] \quad \dfrac{[\neg\exists xPx \land \neg\exists yQy]}{\neg\exists xPx}}{\exists yQy} \quad \dfrac{[\exists yQy] \quad \dfrac{[\neg\exists xPx \land \neg\exists yQy]}{\neg\exists yQy}}{\exists xPx}}{\exists xPx \leftrightarrow \exists yQy}}{\exists xPx \leftrightarrow \exists yQy}$$

6.2(vi)

$$\dfrac{[\forall xPx] \quad \dfrac{\dfrac{\dfrac{[\forall xPx]}{Pa} \quad \dfrac{\forall x(Px \to \exists y(Ryx \land Py))}{Pa \to \exists y(Rya \land Py)}}{\exists y(Rya \land Py)} \quad \dfrac{\dfrac{\dfrac{[\forall xPx]}{Pb} \quad \dfrac{\forall x(Px \to \exists y(Ryx \land Py))}{Pb \to \exists y(Ryb \land Py)}}{\exists y(Ryb \land Py)} \quad \dfrac{\dfrac{[\forall xPx]}{Pc} \quad \dfrac{\dfrac{[Rcb \land Pc]}{[Rcb]} \quad \dfrac{\dfrac{[Rba \land Pb]}{[Rba]} \quad \dfrac{\dfrac{\dfrac{\forall x\forall y\forall z(Ryx \to (Rzy \to \neg Pz))}{\forall y\forall z(Rya \to (Rzy \to \neg Pz))}}{\forall z(Rba \to (Rzb \to \neg Pz))}}{Rba \to (Rcb \to \neg Pc)}}{Rcb \to \neg Pc}}{\neg Pc}}{\neg\forall xPx}}{\neg\forall xPx}}{\neg\forall xPx}}{\neg\forall xPx}$$

7 Formalisation in Predicate Logic

Exercise 7.1. Show that the following argument is logically valid by formalizing it in the language $\mathcal{L}_2$ of predicate logic and establishing the validity of the formalized argument by means of a proof in Natural Deduction.

> Every villager is the descendant of a villager. Kate is a villager. Therefore Kate is the descendant of a villager, who is the descendant of some villager as well.

Exercise 7.2. Reveal the ambiguities in the following sentences by formalizing them in two or more different ways in $\mathcal{L}_2$.

(i) Some property is instantiated by every object.
(ii) The chaplain married John's sister.
(iii) Fiona will have to postpone her trip and she will need to apply for special permission if the documents don't arrive in time.
(iv) All that glitters is not gold.

Exercise 7.3. Formalize each of the following sentences in as much detail as possible, explicitly specifying your dictionary. If there are any non-extensional expressions in the English sentence, demonstrate their failure of extensionality using examples, and explain what this means for the formalization.

(i) Every writer admires the author of *The Man Without Qualitites*.
(ii) Every writer admires some writer.
(iii) Jane hopes that she will see her brother tomorrow, but she believes it is likely that she won't.
(iv) For any atom of polonium-218 and atom of uranium-238, the former is more likely to decay in the next five minutes than the latter.

Exercise 7.4. Recall that a set of $\mathcal{L}_2$-sentences Γ is defined to be syntactically consistent if and only if there is an $\mathcal{L}_2$-sentence ϕ such that $\Gamma \nvdash \phi$. Prove that Γ is syntactically consistent if and only if $\Gamma \nvdash P \wedge \neg P$.

Exercise 7.5. Determine whether the following sentence is logically true in predicate logic:

> There is someone such that, if he or she is asleep, everyone is asleep.

Exercise 7.6. Show that if ϕ is a tautology in the language $\mathcal{L}_1$ of propositional logic, then there are infinitely many proofs of ϕ in the system of Natural Deduction.

Exercise 7.7. For any natural number $n \geq 1$, specify an inconsistent set Γ_n of exactly n $\mathcal{L}_1$-sentences such that all sets of sentences in Γ_n not containing all sentences in Γ_n are consistent. Sketch an argument showing that the specified set satisfies these conditions.

Answer to 7.1. I use the following dictionary:

P: … is a villager
R: … is a descendant of …
a: Kate

With this, the argument can be formalized as the following valid argument of $\mathcal{L}_2$:
$\forall x(Px \to \exists y(Py \wedge Rxy)), Pa \vdash \exists x(Px \wedge Rax \wedge \exists y(Py \wedge Rxy))$

$$\dfrac{\dfrac{Pa \quad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}}{\exists y(Py \wedge Ray)} \qquad \dfrac{\dfrac{[Pb \wedge Rab] \quad \dfrac{\dfrac{[Pb \wedge Rab]}{Pb} \quad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pb \to \exists y(Py \wedge Rby)}}{\exists y(Py \wedge Rby)}}{Pb \wedge Rab \wedge \exists y(Py \wedge Rby)}}{\exists x(Px \wedge Rax \wedge \exists y(Py \wedge Rxy))}}{\exists x(Px \wedge Rax \wedge \exists y(Py \wedge Rxy))}$$

Answer to 7.2.

(i) I use the following dictionary:

P: … is a property
Q: … is an object
R: … is instantiated by …

There are two ways of reading the sentence, which can be formalized in the following two ways:

$$\exists x(Px \wedge \forall y(Qy \to Rxy))$$
$$\forall y(Qy \to \exists x(Px \wedge Rxy))$$

(ii) The ambiguity in this sentence is not structural, but lexical: the verb "to marry" has two meanings; it can either mean to enter into a marriage or to perform the ceremony of marriage. The corresponding two ways of reading the sentence can therefore be formalized using the same $\mathcal{L}_2$-sentence Pab; the difference in formalization consists in the fact that we can use either of the following two dictionaries:

P: … marries (in the sense of entering into a marriage with) …
a: the chaplain
b: John's sister

or

P: … marries (in the sense of performing the ceremony of marriage) …
a: the chaplain
b: John's sister

(iii) I use the following dictionary:

P: Fiona will have to postpone her trip
Q: the documents don't arrive in time
R: Fiona will need to apply for special permission

There are two ways of reading the sentence, which can be formalized in the following two ways:

$$P \land (Q \to R)$$

$$Q \to P \land R$$

(iv) I use the following dictionary:

P: … glitters
Q: … is gold

There are two ways of reading the sentence, which can be formalized in the following two ways:

$$\forall x(Px \to \neg Qx)$$

$$\neg\forall x(Px \to Qx)$$

Answer to 7.3.

(i) $\forall x(Px \to Qx)$

P: … is a writer
Q: … admires the author of *The Man Without Qualitites*

"… admires …" is not extensional: Lois Lane may admire Superman without admiring Clark Kent, even though Clark Kent is Superman. Therefore, "admires" does not express a binary relation, and so we can't formalize it using a two-place predicate letter.

(ii) $\forall x(Px \to Qx)$

P: … is a writer
Q: … admires some writer.

As in (i), "… admires …" is not extensional, so we can't formalize it using a two-place predicate letter. Therefore, we are also unable to formalize "some writer" using an existential quantifier.

(iii) $Pa \wedge Qa$

P: … hopes that she will see her brother tomorrow
Q: … believes it is likely that she won't see her brother tomorrow
a: Jane

Neither "… hopes that she will see … tomorrow" nor "… believes it is likely that she won't see … tomorrow" is extensional. We may imagine an astronomer incorrectly thinking that Hesperus and Phosphorus are distinct celestial bodies, and consequently hoping that she will see Hesperus tomorrow without thinking that she will see Phosphorus tomorrow; likewise, she might believe that it is likely that she won't see Hesperus tomorrow without believing it to be likely that she won't see Phosphorus tomorrow.

(iv) $\forall x \forall y(Px \wedge Qy \to Rxy)$

P: … is an atom of polonium-218
Q: … is an atom of uranium-238
R: … is more likely to decay in the next five minutes than…

This formalization assumes that "likely" is read as expressing objective chances, which is natural, as that the sentence is about nuclear decay. If it is read as expressing our subjective credences, then "… is more likely to decay in the next five minutes than …" is not extensional (since we might well be confused about the identities of certain atoms). On this reading, the sentence has to be formalized using a single sentence letter.

Answer to 7.4. If $\Gamma \nvdash P \wedge \neg P$, then there is an $\mathcal{L}_2$-sentence ϕ such that $\Gamma \nvdash \phi$, namely $P \wedge \neg P$, so Γ is syntactically consistent. If $\Gamma \vdash P \wedge \neg P$, then we can extend any proof of this to a proof of $\Gamma \vdash P$ and to a proof of $\Gamma \vdash \neg P$, using conjunction elimination. Using negation elimination, we can then combine the resulting proof to a proof of $\Gamma \vdash \varphi$, for any $\mathcal{L}_2$-sentence φ. So Γ is not syntactically consistent.

Answer to 7.5. The sentence can be formalized as follows: $\exists x(Px \to \forall y Py)$

P: … is asleep

This $\mathcal{L}_2$-sentence is logically true. We can establish this in the Natural Deduction system using the following proof, where dots indicate a proof of $\exists y \neg Py \vee \neg\exists y \neg Py$ along the lines of Example 6.6 of the *Logic Manual*:

$$\dfrac{\begin{array}{c}\vdots\\ \exists y\neg Py \vee \neg\exists y\neg Py\end{array} \quad \dfrac{[\exists y\neg Py] \quad \dfrac{\dfrac{\dfrac{[Pa]\ [\neg Pa]}{\forall y Py}}{Pa \to \forall y Py}}{\exists x(Px \to \forall y Py)}}{\exists x(Px \to \forall y Py)} \quad \dfrac{\dfrac{\dfrac{\dfrac{[\neg\exists y\neg Py] \quad \dfrac{[\neg Pa]}{\exists y\neg Py}}{Pa}}{\forall y Py}}{Pa \to \forall y Py}}{\exists x(Px \to \forall y Py)}}{\exists x(Px \to \forall y Py)}$$

Answer to 7.6. By completeness (Theorem 7.1 in the *Logic Manual*), if ϕ is a tautology, there is a proof of ϕ in the system of Natural Deduction. Consider any such proof:

$$\begin{array}{c} \vdots \\ \phi \end{array}$$

Given such a proof, the result of deriving $\phi \wedge \phi$ from two copies of it is a proof of $\phi \wedge \phi$, from which we can derive ϕ again by conjunction elimination:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \phi\end{array} \quad \begin{array}{c}\vdots \\ \phi\end{array}}{\phi \wedge \phi}}{\phi}$$

This procedure allows us to turn any proof of ϕ into a more complex proof of ϕ. So, starting from any proof of ϕ, we can apply the procedure, resulting in a more complex proof, apply the procedure to this second proof, obtaining an even more complex third proof, etc. for all natural numbers n. So for every natural number n, we obtain a distinct proof of ϕ from our original proof. Since there are infinitely many natural numbers, there are infinitely many proofs of ϕ in the system of Natural Deduction.

Answer to 7.7. Let Γ_n be the set containing the sentence $P_1 \wedge \neg P_{n+1} \wedge P_i \rightarrow P_{i+1}$ for every natural number i such that $1 \leq i \leq n$.

We first argue semantically that Γ_n is inconsistent. Any $\mathcal{L}_1$-structure satisfying Γ_n would have to make P_1 true and P_{n+1} false, but it would also have to make $P_i \rightarrow P_{i+1}$ true for every natural number i such that $1 \leq i \leq n$, which is impossible.

We now argue that any set of sentences in Γ_n not containing all sentences in Γ_n is consistent. For any such set, there is a natural number i between 1 and n such that the set does not contain the sentence $P_1 \wedge \neg P_{n+1} \wedge P_i \rightarrow P_{i+1}$. Consider an $\mathcal{L}_1$-structure in which for all natural numbers j, the sentence letter P_j is true if and only if $j \leq i$. Then $P_1 \wedge \neg P_{n+1} \wedge P_j \rightarrow P_{j+1}$ will be true in the structure for all natural numbers j between 1 and n apart from i, and so in particular all sentences in the set under consideration will be true.

8 Identity and Definite Descriptions

Exercise 8.1. Formalize the following sentences in the language $\mathcal{L}_=$ of predicate logic with identity in as much detail as possible, explicitly specifying your dictionary.

(i) There are exactly three cats in the room.
(ii) If Walter Scott is the author of Waverley and Walter Scott is the author of Ivanhoe, then the author of Waverley is the author of Ivanhoe.

Exercise 8.2. Refute each of the following claims by means of a counterexample.

(i) $\neg a = b \land \neg b = c \vdash \neg a = c$
(ii) $\exists x \exists y (Ryx \land \neg x = y), \forall x \forall y \forall z (Rzy \to (Ryx \to Rzx)) \vdash$
$\exists x \exists y \exists z (\neg x = y \land \neg y = z \land \neg x = z)$
(iii) $\forall x \forall y \forall z (Rxy \land Rxz \to y = z) \vdash \forall x \forall y \forall z (Ryx \land Rzx \to y = z)$

Exercise 8.3. Establish the following claims by means of proofs in Natural Deduction.

(i) $\vdash \forall x \forall y \forall z (x = y \land y = z \to x = z)$
(ii) $\forall x \forall y (Rxy \to \neg Ryx) \vdash \forall x \forall y (Rxy \land Ryx \to x = y)$
(iii) $\forall x \forall y (\forall z (Rzx \leftrightarrow Rzy) \to x{=}y), \exists x \forall y \neg Ryx \vdash \exists x \forall y (\forall z \neg Rzy \leftrightarrow x{=}y)$

Exercise 8.4. Determine for each of the following sentences whether it is logically true by suitably formalizing it in the language $\mathcal{L}_=$ of predicate logic with identity, specifying a dictionary, and proving the resulting sentence in the system of Natural Deduction or providing a counterexample.

(i) The set with no elements doesn't contain any elements.

(ii) If there are more than two white elephants, then it is not the case that there are white elephants in India.
(iii) If the greatest Roman orator is Cicero, and Tully is Cicero, then Tully is an orator.

Exercise 8.5. (a) Compare and contrast to what extent these sentences may be formalized in $\mathcal{L}_1$.

(i) On Wednesday we will play football or basketball.
(ii) On Wednesday we can play football or basketball.

(b) Compare and contrast to what extent these sentences may be formalized in $\mathcal{L}_2$.

(i) Every student answered an easy question.
(ii) Most students answered an easy question.

(c) Compare and contrast to what extent these sentences may be formalized in $\mathcal{L}_=$.

(i) The best student answered three questions.
(ii) The average student answered three questions.

Exercise 8.6. (a) Formalize the following as a valid argument in $\mathcal{L}_2$, using the dictionary below. Demonstrate the validity of your formalization using Natural Deduction.

Q: Matter is atomless
P^2: … is a proper part of …
R^2: … is smaller than …
a: the smallest object

Matter is not atomless. Assuming, on the contrary, that it is, then every object has a proper part. A proper part of anything is smaller than it is. If something is smaller than the smallest object, then something is smaller than everything. But this is impossible: nothing is smaller than itself.

(b) Formalize the argument in part (a) once more, this time as an *invalid* argument in $\mathcal{L}_=$, using the same dictionary but without the constant a. Demonstrate the invalidity of your formalization by specifying a counterexample.
(c) What is it that makes the $\mathcal{L}_2$ formalization valid when the $\mathcal{L}_=$ formalization is not valid?

Exercise 8.7. Recall the compactness theorem for propositional logic, which was mentioned on p. 42 of the *Logic Manual*: If a sentence ϕ of $\mathcal{L}_1$ follows from a set of sentences Γ of $\mathcal{L}_1$, then there is a *finite* set Δ of sentences in Γ from which ϕ follows.

There are analogous compactness theorems for predicate logic and predicate logic with identity: If a sentence ϕ of $\mathcal{L}_2/\mathcal{L}_=$ follows from a set of sentences Γ of $\mathcal{L}_2/\mathcal{L}_=$, then there is a *finite* set Δ of sentences in Γ from which ϕ follows.

(a) Show that for each of the three logics, the compactness theorem follows from the adequacy theorem (Theorems 6.10, 7.3 and 8.7 in the *Logic Manual*). To do so, make use of the fact that Natural Deduction proofs are finite, in the sense that every such proof contains only a finite number of occurrences of formulas.
(b) Specify a set Γ of sentences of $\mathcal{L}_=$ such that for all $\mathcal{L}_2$-structures $\mathcal{A}$, all elements of Γ are true in $\mathcal{A}$ if and only if the domain of $\mathcal{A}$ is infinite.
(c) Use the compactness theorem for $\mathcal{L}_=$ established in (a) and the set specified in (b) to show that there is no *single* sentence of $\mathcal{L}_=$ which is true in an $\mathcal{L}_2$-structure if and only if its domain is infinite.

Answer to 8.1.

(i) $\exists x \exists y \exists z(Px \wedge Qx \wedge Py \wedge Qy \wedge Pz \wedge Qz \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge \forall x_1(Px_1 \wedge Qx_1 \rightarrow x_1 = x \vee x_1 = y \vee x_1 = z))$

P^1: … is a cat
Q^1: … is in the room

(ii) $\forall x(Pxa \leftrightarrow x = c) \wedge \forall x(Pxb \leftrightarrow x = c) \rightarrow$
$\exists x \exists y(\forall z(Pza \leftrightarrow z = x) \wedge \forall z(Pzb \leftrightarrow z = y) \wedge x = y)$

P^2: … authored …
a: Waverley
b: Ivanhoe
c: Walter Scott

Answer to 8.2.

(i) Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |a|_{\mathcal{A}} &= 0 \\ |b|_{\mathcal{A}} &= 1 \\ |c|_{\mathcal{A}} &= 0 \end{aligned}$$

(ii) Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |R|_{\mathcal{A}} &= \{\langle 0,1\rangle\} \end{aligned}$$

(iii) Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |R|_{\mathcal{A}} &= \{\langle 0,1\rangle, \langle 1,1\rangle\} \end{aligned}$$

Answer to 8.3. (i) $\vdash \forall x\forall y\forall z(x = y \land y = z \to x = z)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a = b \land b = c]}{a = b} \qquad \dfrac{[a = b \land b = c]}{b = c}}{a = c}}{a = b \land b = c \to a = c}}{\forall z(a = b \land b = z \to a = z)}}{\forall y\forall z(a = y \land y = z \to a = z)}}{\forall x\forall y\forall z(x = y \land y = z \to x = z)}$$

(ii) $\forall x\forall y(Rxy \to \neg Ryx) \vdash \forall x\forall y(Rxy \land Ryx \to x = y)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Rab \land Rba]}{Rba} \qquad \dfrac{\dfrac{[Rab \land Rba]}{Rab} \qquad \dfrac{\dfrac{\forall x\forall y(Rxy \to \neg Ryx)}{\forall y(Ray \to \neg Rya)}}{Rab \to \neg Rba}}{\neg Rba}}{a = b}}{Rab \land Rba \to a = b}}{\forall y(Ray \land Rya \to a = y)}}{\forall x\forall y(Rxy \land Ryx \to x = y)}$$

(iii) For reasons of space, the proof of of $\forall x\forall y(\forall z(Rzx \leftrightarrow Rzy) \to x = y), \exists x\forall y\neg Ryx \vdash \exists x\forall y(\forall z\neg Rzy \leftrightarrow x = y)$ is given on the last page.

Answer to 8.4.

(i) This sentence is not logically true. We can formalize it as follows:
$\exists x(\forall y(Py \land \neg\exists zQzy \leftrightarrow y = x) \land \neg\exists zQzx)$

P^1: … is a set
Q^2: … is an element of …

For a counterexample, let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$D_{\mathcal{A}} = \{0\}$$
$$|P|_{\mathcal{A}} = \varnothing$$
$$|Q|_{\mathcal{A}} = \varnothing$$

(ii) This sentence is not logically true. We can formalize it as follows:
$\exists x \exists y \exists z(Px \land Py \land Pz \land \neg x = y \land \neg x = z \land \neg y = z) \to \neg \exists x(Px \land Qx)$

P^1: … is a white elephant
Q^1: … is in India

For a counterexample, let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$D_{\mathcal{A}} = \{0,1,2\}$$
$$|P|_{\mathcal{A}} = \{0,1,2\}$$
$$|Q|_{\mathcal{A}} = \{0,1,2\}$$

(iii) This sentence is not logically true. We can formalize it as follows:
$\forall x(Px \land \forall y(Py \land \neg x = y \to Qxy) \leftrightarrow x = a) \land b = a \to Rb$

P^1: … is Roman
Q^2: … is a greater orator than …
R^1: … is an orator
a: Cicero
b: Tully

For a counterexample, let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$D_{\mathcal{A}} = \{0\}$$
$$|P|_{\mathcal{A}} = \{0\}$$
$$|Q|_{\mathcal{A}} = \varnothing$$
$$|R|_{\mathcal{A}} = \varnothing$$
$$|a|_{\mathcal{A}} = 0$$
$$|b|_{\mathcal{A}} = 0$$

Answer to 8.5.

(a) (i) can be formalized as a disjunction, but (ii) has to be formalized as a single sentence letter. To see that (ii) cannot be understood as a disjunction, note that if on Wednesday, we can play football, then the disjunction of the claim that on Wednesday, we can play football, and the claim that on

Wednesday, we can play basketball, is true. However, this does not mean that (ii) is true, since the claims just stated don't entail that on Wednesday, we can play basketball, which is guaranteed by (ii).
So (i) can be formalized as $P \vee Q$ and (ii) as R, using the following dictionary:

P: On Wednesday we will play football
Q: On Wednesday we will play basketball
R: On Wednesday we can play football or basketball.

(b) Since $\mathcal{L}_2$ does not have a representation of "most", (ii) has to be represented by a single sentence letter. Using universal and existential quantifiers, we can give a more detailed formalization of (i). With the following dictionary, (i) can be formalized as $\forall x(Px \to \exists y(Qy \wedge Rxy))$ and (ii) as P_1:

P^1: … is a student
Q^1: … is an easy question
R^2: … answered …
P_1: Most students answered an easy question.

(c) Using quantifiers and identity, we can give a detailed formalization of (i). We cannot do the same for (ii), since "the average student" does not refer to a particular student, and so can't be represented by a constant, nor can it be given a complex formalization in terms of quantifiers and identity. Therefore (ii) has to be formalized using a single sentence letter. Using the following dictionary, (i) can be formalized as $\exists x(\forall y(\forall z(\neg z = y \to Pyz) \leftrightarrow y = x) \wedge \exists y_1 \exists y_2 \exists y_3(Qy_1 \wedge Qy_2 \wedge Qy_3 \wedge \neg y_1 = y_2 \wedge \neg y_1 = y_3 \wedge \neg y_2 = y_3 \wedge Rxy_1 \wedge Rxy_2 \wedge Rxy_3))$, and (ii) as P_1.

P^2: … is a better student than …
Q^1: … is a question
R^2: … answered …
P_1: The average student answered three questions.

Answer to 8.6.

(a) $Q \to \forall x\exists yPyx, \forall x\forall y(Pxy \to Rxy), \exists xRxa \to \exists x\forall yRxy, \neg\exists xRxx \vdash \neg Q$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Q] \quad Q\to\forall x\exists yPyx}{\forall x\exists yPyx}}{\exists yPya} \quad \dfrac{\dfrac{[Pba] \quad \dfrac{\dfrac{\forall x\forall y(Pxy\to Rxy)}{\forall y(Pby\to Rby)}}{Pba\to Rba}}{Rba}}{\exists xRxa}}{\exists xRxa} \quad \exists xRxa\to\exists x\forall yRxy}{\exists x\forall yRxy} \quad \dfrac{\dfrac{[\forall yRcy]}{Rcc}}{\exists xRxx}}{\exists xRxx} \quad \neg\exists xRxx}{\neg Q}$$

(b) $Q \to \forall x\exists yPyx, \forall x\forall y(Pxy \to Rxy), \exists x(\forall y(\forall z(\neg z = y \to Ryz) \leftrightarrow x = y) \land \exists yRyx) \to \exists x\forall yRxy, \neg\exists xRxx \vdash \neg Q$

For a counterexample, let $\mathcal{A}$ be an $\mathcal{L}_2$-structure such that:

$$\begin{aligned} D_{\mathcal{A}} &= \{0,1\} \\ |P|_{\mathcal{A}} &= \{\langle 0,1\rangle, \langle 1,0\rangle\} \\ |Q|_{\mathcal{A}} &= \mathrm{T} \\ |R|_{\mathcal{A}} &= \{\langle 0,1\rangle, \langle 1,0\rangle\} \end{aligned}$$

(c) In the first formalization, “the smallest object” is formalized as a constant. In $\mathcal{L}_2$, all constants denote some element of the domain, so the use of a constant in the formalization implicitly introduces the presupposition in the formalized argument that there is a smallest object. No such assumption – implicit or explicit – is present in the second formalization, which allows counterexamples in which there is no smallest object.

Answer to 8.7.

(a) Consider any of the three logics, and let ϕ be a sentence and Γ a set of sentences of the relevant language. Assume ϕ follows from Γ, i.e., $\Gamma \vDash \phi$. By the adequacy theorem for the logic under consideration, it follows that

$\Gamma \vdash \phi$, so there is a proof of ϕ with only sentences in Γ as undischarged assumptions. As noted in the exercise, this proof can only contain a finite number of occurrences of formulas. So the set Δ of sentences in Γ which also occur as undischarged assumptions in this proof is finite. Since Δ is the set of undischarged assumptions in our proof, it follows that $\Delta \vdash \phi$, and so via the adequacy theorem again that $\Delta \vDash \phi$. This concludes the proof: Δ is finite set of sentences in Γ from which ϕ follows.

(b) Let Γ be a set of sentences, one for each natural number n, which say that there are (at least) n objects. More precisely, we can define this set as follows: For each natural number n, define η_n to be a sentence which is true in an $\mathcal{L}_2$-structure if and only if its domain contains at least n elements. E.g., let:

$$\begin{aligned}
\eta_1 &= \exists x_1\, x_1 = x_1 \\
\eta_2 &= \exists x_1 \exists x_2\, \neg x_1 = x_2 \\
\eta_3 &= \exists x_1 \exists x_2 \exists x_3 (\neg x_1 = x_2 \wedge \neg x_1 = x_3 \wedge \neg x_2 = x_3) \\
&\vdots
\end{aligned}$$

Let Γ be the set of sentences η_n for every natural number n, i.e., $\Gamma = \{\eta_n : n \text{ is a natural number}\}$.

(c) Assume for contradiction that there is a sentence ϕ of $\mathcal{L}_=$ which is true in an $\mathcal{L}_2$-structure if and only if its domain is infinite. Then using the set Γ from (b), $\Gamma \vDash \phi$. So by the compactness theorem for $\mathcal{L}_=$, established in (a), there is a finite set Δ of sentences in Γ such that $\Delta \vDash \phi$. We distinguish two cases:

Case 1: Δ is empty. Then every element of Δ is true in all finite $\mathcal{L}_2$-structures, but ϕ is by assumption false in all such structures.

Case 2: Δ is not empty. Then there is a largest natural number n such that η_n is an element of Δ (recall that Δ is finite). So in any finite $\mathcal{L}_2$-structure whose domain contains at least n elements, all elements of Δ are true and ϕ is false.

So in both cases, there are $\mathcal{L}_2$-structures in which all elements of Δ are true and ϕ is false. Therefore $\Delta \nvDash \phi$, contradicting what we have established. So there is no sentence ϕ of $\mathcal{L}_=$ which is true in an $\mathcal{L}_2$-structure if and only if its domain is infinite.

8.3(iii)

$$
\dfrac{\exists x \forall y \neg Ryx \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Rca] \qquad \dfrac{[\forall y \neg Rya]}{\neg Rca}}{Rcb} \qquad \dfrac{[Rcb] \qquad \dfrac{[\forall z \neg Rzb]}{\neg Rcb}}{Rca}}{Rca \leftrightarrow Rcb}}{\forall z(Rza \leftrightarrow Rzb)} \qquad \dfrac{\dfrac{\forall x \forall y(\forall z(Rzx \leftrightarrow Rzy) \rightarrow x = y)}{\forall y(\forall z(Rza \leftrightarrow Rzy) \rightarrow a = y)}}{\forall z(Rza \leftrightarrow Rzb) \rightarrow a = b}}{a = b} \qquad \dfrac{\dfrac{\dfrac{[\forall y \neg Rya] \qquad [a = b]}{\forall y \neg Ryb}}{\neg Rcb}}{\forall z \neg Rzb}}{\forall z \neg Rzb \leftrightarrow a = b}}{\dfrac{\forall y(\forall z \neg Rzy \leftrightarrow a = y)}{\exists x \forall y(\forall z \neg Rzy \leftrightarrow x = y)}}}{\exists x \forall y(\forall z \neg Rzy \leftrightarrow x = y)}
$$