Corso di Laurea in Informatica

Prova scritta di Analisi Matematica (I Modulo)

9 Settembre 2022 (M.Mughetti)

Risolvere gli esercizi seguenti, scrivendo e motivando dettagliatamente il procedimento seguito. Soluzioni prive di calcoli e spiegazioni NON SARANNO VALUTATE.

Esercizio 1
Sia data la funzione $f : \mathcal{D}(f) \to \mathbf{R}$

$$f(x) = \frac{x^2 + 3x + 4}{x^2 - 1}.$$

I. Disegnare il suo grafico.

II. Calcolare l'immagine di f sul suo dominio naturale $\mathcal{D}(f)$.

III. Stabilire per quali $\lambda \in \mathbf{R}$ l'equazione $f(x) = \lambda$ ha un'unica soluzione.

Esercizio 2
Sapendo che, per $t \to 0$,

$$\begin{aligned} \ln(1+t) &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + o(t^4) \\ e^t &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + o(t^5) \end{aligned}$$

Calcolare

$$\lim_{x \to 0} \frac{(1+x)^{x+1} - e^{x^2} - x}{x^3}$$

Risposta:
CALCOLARE, prima gli sviluppi di Taylor di $(1+x)^{x+1}, e^{x^2}$, **NELLA FORMA** in cui saranno usati nel limite dato; infine risolvere il limite assegnato.

ANALISI MATEMATICA. SECONDO MODULO
CDS INFORMATICA
9 SETTEMBRE 2022

1. Individuare e classificare i punti critici della funzione

$$f(x,y) = x^3 - x^2y + y^2 - 2y.$$

2. Sull'insieme $A = \left\{(x,y) \in \mathbb{R}^2 : |x| \leq y \leq 1 - \frac{|x|}{2}\right\}$ calcolare l'integrale

$$\int_A \sqrt{x+y}\; dxdy.$$

3. Scrivere

1

$$f'(x) = \left(\frac{x^2+3x-4}{x^2-1}\right)' = \frac{(2x+3)(x^2-1) - 2x(x^2+3x+4)}{(x^2-1)^2} =$$

$$= \frac{\cancel{2x^3} + 3x^2 - 2x - 3 - \cancel{2x^3} - 6x^2 - 8x}{(x^2-1)^2} = \frac{-3x^2 - 10x - 3}{(x^2-2)^2} =$$

$$= \frac{\overbrace{(-x-3)}^{\mathrm{I}}\ \overbrace{(3x+1)}^{\mathrm{II}}}{\underbrace{(x^2-1)^2}_{\mathrm{III}}}$$

		-3		-1		$-\frac{1}{3}$		1	
I	+	0	−		−		−		−
II	−		−		−	0	+		+
III	+		+	×	+		+	×	+
	↘		↗		↗		↘		↘

$x = -3, -\frac{1}{3}$ max loc.

$x = -1, 1$ min loc.

$$\lim_{x \to -\infty} \frac{x^2+3x+4}{x^2-1} = \frac{\infty - \infty}{\infty} = \lim_{x \to -\infty} \frac{\cancel{x^2}\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)}{\cancel{x^2}\left(1 - \frac{1}{x^2}\right)} = 1$$

$$f(-3) = \frac{9 - 9 + 4}{9 - 1} = \frac{4}{8} = \frac{1}{2}$$

$$\lim_{x \to -1^-} \frac{x^2+3x+4}{x^2-1} = \frac{1 - 3 + 4}{0^+} = +\infty$$

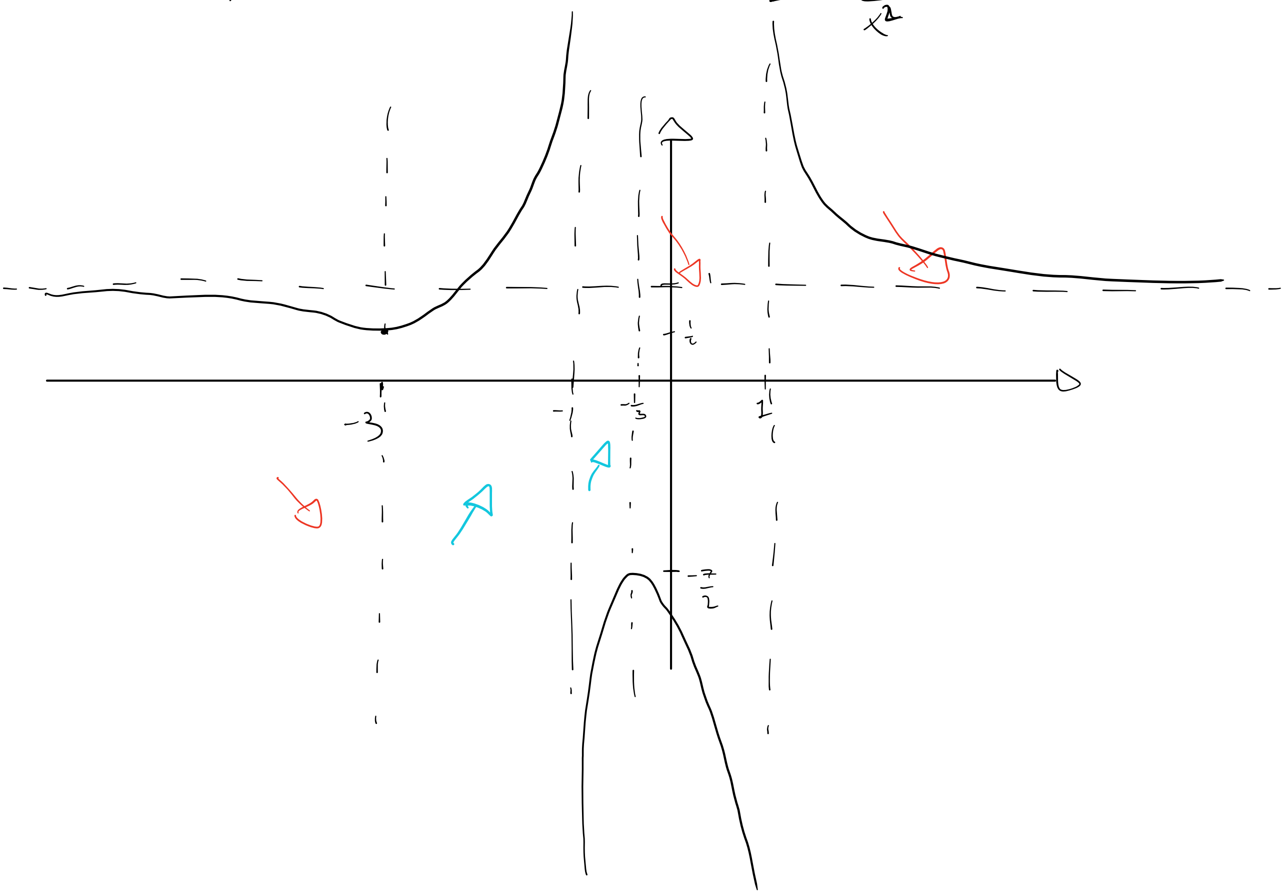
$$\lim_{x \to -1^+} \frac{x^2+3x+4}{x^2-1} = \frac{1 - 3 + 4}{0^-} = -\infty$$

$$f\left(-\frac{1}{3}\right) = \frac{\frac{1}{9} - 1 + 4}{\frac{1}{9} - 1} = \frac{\frac{1+27}{3}}{-\frac{8}{9}} = \frac{\frac{28}{9}}{-\frac{8}{9}} = -\frac{\cancel{28}^{7}}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{8}_{2}} = -\frac{7}{2}$$

$$\lim_{x \to 1^-} \frac{x^2+3x+4}{x^2-1} = \frac{1+3+4}{0^-} = -\infty$$

$$\lim_{x \to 1^+} \frac{x^2+3x+4}{x^2-1} = \frac{1+3+4}{0^+} = +\infty$$

$$\lim_{x \to +\infty} \frac{x^2+3x+4}{x^2-1} = \dots = \frac{1+\frac{3}{x}+\frac{4}{x^2}}{1-\frac{1}{x^2}} = 1$$



Dominio $x \in \mathbb{R} \setminus \{\pm 1\}$

Immagine $y \in \left]-\infty ; -\frac{7}{2}\right] \cup \left[\frac{1}{2} ; +\infty\right[$

L'equazione $f(x) = \lambda$ ha un'unica soluzione reale per

$\lambda = -\frac{7}{2}, \frac{1}{2}, 1$

2.

$$\lim_{x\to 0} \frac{(x+1)^{x+1} - e^{x^2} - x}{x^3}$$

$$e^t = 1 + t + \frac{1}{2!} t^2 + o(t^2) \qquad t = x^2$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2} x^4 + o(x^4)$$

$$(x+1)^{x+1} = e^{(x+1)\ln x+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$(x+1)\ln(1+x) = x^2 + x - \frac{x^3}{2} - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) =$$

$$= x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)$$

$$e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + o(t^3) \qquad t = (x+1)\ln(x+1)$$

$$e^{(x+1)\ln(x+1)} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{2!}\left(x + \frac{x^2}{2} - \frac{x^3}{6}\right) + \frac{1}{3!}\left(x + \frac{x^2}{2} - \frac{x^3}{6}\right)^3 + o(x^3)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{2}\left(x^2 + x^3\right) + \frac{1}{3!} x^3 + o(x^3)$$

$$= 1 + x + x^2 + \frac{x^3}{2} + o(x^3)$$

$$\lim_{x\to 0} \frac{(1+x)^{1+x} - e^{x^2} - x}{x^3} = \lim_{x\to 0} \frac{\cancel{1} + \cancel{x} + \cancel{x^2} + \frac{x^3}{2} - \cancel{1} - \cancel{x^2} - \cancel{x} + o(x^3)}{x^3} =$$

$$= \lim_{x\to 0} \frac{1}{2} + \frac{o(x^3)}{x^3} = \frac{1}{2}$$

3.

$D_x f = 3x^2 - 2xy$

$D_y f = -x^2 + 2y - 2$

$\nabla f(x,y) = (3x^2 - 2xy, \; -x^2 + 2y - 2)$

$$\begin{cases} x(3x - 2y) = 0 & (1) \\ 2y - x^2 - 2 = 0 & (2) \end{cases}$$

$x = 0 \qquad y = \frac{3}{2}x$

$$\begin{cases} 0 = 0 \\ 2y - 2 = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = 1 \end{cases} \qquad (0,1)$$

$$\begin{cases} y = \frac{3}{2}x \\ -x^2 + 3x - 2 = 0 \end{cases} \qquad \begin{cases} y = \frac{3}{2}x \\ (-x+2)(x-1) \end{cases} \qquad (2,3) \quad (1, \tfrac{3}{2})$$

$D_{xx} f = 6x - 2y$

$D_{xy} f = -2x$

$D_{yx} f = -2x$

$D_{yy} f = +2$

$$H_f \begin{pmatrix} 6x - 2y & -2x \\ -2x & +2 \end{pmatrix}$$

$$H_{f(0,1)} \begin{pmatrix} -2 & 0 \\ 0 & +2 \end{pmatrix} \text{Det}(H_{f(0,1)}) < 0$$

$(0,1)$ punto di sella

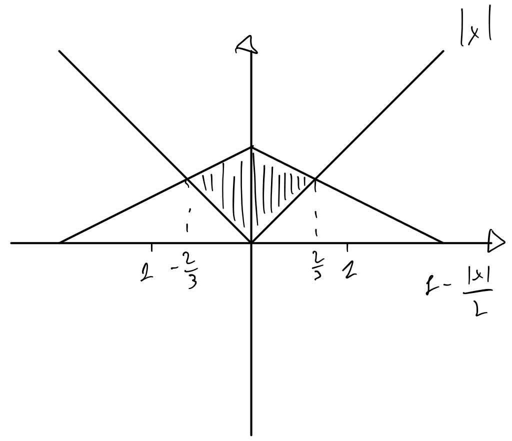
$$H_{f(2,3)} \begin{pmatrix} 6 & -4 \\ -4 & +2 \end{pmatrix} \text{Det}(H_{f(2,3)}) < 0$$

$(2,3)$ punto di sella

$Hf(1, \frac{3}{2}) \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$ $\mathrm{Det}\left(H_{f(1,\frac{3}{2})}\right) > 0$

$(1, \frac{3}{2})$ punto di min rel.

4.



$|x| \le 1 - \frac{|x|}{2} \to |x| < \frac{2}{3}$

$$\int_A \sqrt{x+y}\,dx\,dy = \int_{-\frac{2}{3}}^{\frac{2}{3}} \int_{|x|}^{1-\frac{|x|}{2}} \sqrt{x+y}\,dy\,dx =$$

$$= \frac{2}{3}\int_{-\frac{2}{3}}^{\frac{2}{3}} \left| (x+y)^{\frac{3}{2}} \right|_{|x|}^{1-\frac{|x|}{2}} dx = \frac{2}{3}\int_{-\frac{2}{3}}^{0} \left| (x+y)^{\frac{3}{2}} \right|_{-x}^{1+\frac{x}{2}} dx + \frac{2}{3}\int_{0}^{\frac{2}{3}} \left| (x+y)^{\frac{3}{2}} \right|_{x}^{1-\frac{x}{2}} dx$$

$$= \frac{2}{3}\left(\int_{-\frac{2}{3}}^{0} \left(x+1+\frac{x}{2}\right)^{\frac{3}{2}} - 0^{\frac{3}{2}}\,dx + \int_{0}^{\frac{2}{3}} \left(x+1-\frac{x}{2}\right)^{\frac{3}{2}} - (2x)^{\frac{3}{2}}\,dx \right) =$$

$$= \frac{2}{3}\left(\int_{-\frac{2}{3}}^{0} \left(1+\frac{3}{2}x\right)^{\frac{3}{2}} dx + \int_{0}^{\frac{2}{3}} \left(1+\frac{x}{2}\right)^{\frac{3}{2}} dx - \int_{0}^{\frac{2}{3}} (2x)^{\frac{3}{2}}\,dx \right) =$$

$$= \frac{2}{3}\left(\left| \frac{4}{15}\left(1+\frac{3}{2}x\right)^{\frac{5}{2}} \right|_{-\frac{2}{3}}^{0} + \left| \frac{4}{5}\left(1+\frac{x}{2}\right)^{\frac{5}{2}} - 2^{\frac{3}{2}}\frac{2}{5}(x)^{\frac{5}{2}} \right|_{0}^{\frac{2}{3}} \right) =$$

$$= \frac{2}{3}\left[\left(\frac{4}{15} - \frac{4}{15}\left(\cancel{1-\frac{3}{2}\frac{2}{3}}\right)^{\frac{5}{2}} \right] + \left[\frac{4}{5}\left(1+\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}\right)^{\frac{5}{2}} - 2^{\frac{3}{2}}\cdot\frac{2}{5}\left(\frac{2}{3}\right)^{\frac{5}{2}} + \right.$$

$$\left. - \frac{4}{5} + 0 \right)\Big] = \frac{2}{3}\left(\frac{4}{15} + \frac{4}{5}\cdot\left(\frac{4}{3}\right)^{\frac{5}{2}} - \frac{2^{\frac{5}{2}}}{5}\left(\frac{2}{3}\right)^{\frac{5}{2}} \right) =$$

$$= \frac{2}{3}\left(\frac{4}{15} + \frac{4}{5}\frac{16\cdot 2}{9\sqrt{3}} - \frac{4\sqrt{2}}{5}\cdot\frac{4\sqrt{2}}{9\sqrt{2}} \right) = \frac{2}{3}\left(\frac{4}{15} + \frac{128}{45\sqrt{3}} - \frac{32}{45\sqrt{2}} \right)$$